On a Multiple Item Selling Model with Vector Offers with Applications to Organizational Hiring and a General Sequential Stochastic Assignment Model

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D-L-R Seq. Stoch. Assignment Problem:

\[ N = \{1, \ldots, n\} \] is set of people having values \( p_1, \ldots, p_n \)

Jobs arrive sequentially; job has value \( x \)

return is \( px \)

General Seq. Stoch. Assignment Problem:

\[ N = \{1, \ldots, n\} \]

Jobs arrive sequentially; \( x = (x_1, \ldots, x_n) \)

job can be rejected: \( C, \ 0 < \beta \leq 1 \)

Interpretation; workers for sale; job is a bid.
What if allow bids for multiple workers?
Set of items to sell: \( N = \{1, \ldots, n\} \)

Buyers bid for specified subsets \( S_1, \ldots, S_k \) of items

Bid is a vector \( \mathbf{X} = (X_{S_1}, \ldots, X_{S_k}) \) with known dist.

At most one of the subsets can be sold to each buyer.

State of the system: \( (S, \mathbf{x}) \)

Optimality equation:

\[
V(S, \mathbf{x}) = \max \left( \beta V(S), \max_{1 \leq i \leq k: S_i \subset S} [x_{S_i} + \beta V(S - S_i)] \right) - c
\]

\[
= \max (\beta V(S), R(S, \mathbf{x})) - c
\]

\[
V(T) = E[V(T, \mathbf{X})]
\]

**Proposition 1** \( V(S) \) is the unique value \( v \) such that

\[
c + (1 - \beta)v = E[(R(S, \mathbf{X}) - \beta v)^+] \tag{1}
\]
Numerical Procedure

- Generate iid random offer vectors $X^j, j = 1, \ldots, m$
- Using $R(\{i\}, x) = x_{\{i\}}$, determine $V(i), i = 1, \ldots, n$
- Note that this yields $R(S, x)$ for $|S| = 2$
- Determine $V(S')$ for all two point sets $S$

How to determine $V(S)$ when know $V(T)$ for $T \subset S$

binary search

$$E[(R(S, X) - \beta v^*)^+] \approx \sum_{j=1}^{m} \frac{1}{m} (R(S, X^j) - \beta v^*)^+.$$ 

Proposition 2

$$V(S) \geq \max_{1 \leq i \leq k: S_i \subset S} [E[X_{S_i}] + \beta V(S - S_i)] - c$$
1 A Special Case Model where Buyers bid for all Subsets

Offer vector $Y_1, \ldots, Y_n$
Buyer willing to buy any set $T$ for the price $\Sigma_{i \in T} Y_i$.

$$V(S, y) = \max_{\emptyset \subset S' \subset S} [\sum_{j \in S'} y_j + \beta V(S - S')] - c$$
$$= \max (\beta V(S), R(S, y)) - c$$

B-F gave OE, but not its solution.

Let $\alpha_i(c) \equiv \beta V(\{i\})$.

**Proposition 3** Optimal policy never sells $i$ for the offered value $y_i < \alpha_i(c)$.

Should you always sell $i$ in state $(S, y)$ if $y_i > \alpha_i(c/|S|)$?

**Example:** $n = 2$, $\beta = c = 1$. Suppose $Y_1, Y_2$ ind.

$$P(Y_1 = 1) = .99, \quad P(Y_1 = 10) = .01$$
$$P(Y_2 = 1) = 1 - 10^{-10}, \quad P(Y_2 = 10^{20}) = 10^{-10}$$
Proposition 4 It is optimal in state \((S, y)\) to sell all items in \(S\) when \(y_i \geq \alpha_i(c/|S|)\) for all \(i \in S\).

Proposition 5 For \(|S| \geq 2\),

\[
\max_{i \in S} \{\alpha_i(c)/\beta + V(S - i)\} \leq V(S) \leq \sum_{i \in S} \alpha_i(c/|S|)/\beta
\]
If it is optimal to sell item $1 \in S$ when the offer vector is $x_1, \ldots, x_n$ it is necessarily optimal to sell item 1 if the offer vector were $y_1, \ldots, y_n$ whenever $y_1 > x_1$?

$$P(Y_1 = 1) = .98, P(Y_1 = 2) = .01, P(Y_1 = 10) = .01$$

Optimal to sell item 1 (and item 2) if the offer vector were $(1, 10^{20})$ but optimal to sell neither if the offer vector were $(2, 1)$.

However, if optimal to sell 1 when the offer vector is $x_1, \ldots, x_n$ then optimal to sell 1 if the offer vector were $y_1, \ldots, y_n$ provided that $y_i \geq x_i$ for all $i \in S$.

That is, optimal set to sell is increasing function of offer vector.
Lemma 1 (Bruss-Ferguson)

\[ V(S \cup T) + V(S \cap T) \geq V(S) + V(T) \]

New Proof:
Seller 1 has \(|S| + |T|\) items
arranges in 2 collections: \(S\) and \(T\)

Seller 2 has same \(|S| + |T|\) items
arranges in 2 collections: \(S \cup T\) and \(ST\)

offer vector \(y_1, \ldots, y_n\), buyer will buy any number
sellers cost per period: \(c\) per unsold collection
seller 1 uses opt. policy: yields \(V(S) + V(T)\)
seller 2 matches 1 always choosing from \(ST\) collection
**Proposition 6** In state \((S, y)\), the optimal set
(a) sold is an increasing function of \(y\).
(b) not sold is an increasing function of \(S\).

**A Heuristic Policy when \(n\) is Large**
Problem with \(2n\) items. Randomly partition into two sets \(N_1\) and \(N_2\) of \(n\) items each. Use optimal policies in subproblems with per period cost \(c/2\). Recombine when possible.

**Example 2.** \(n = 2\), \(X_1, X_2\) are ind \((0, 1)\), \(\beta = 1\).
Table 1: For $n = 2$

<table>
<thead>
<tr>
<th>$c$</th>
<th>expected return from heuristic policy</th>
<th>optimal expected return</th>
</tr>
</thead>
<tbody>
<tr>
<td>.1</td>
<td>1.271</td>
<td>1.273</td>
</tr>
<tr>
<td>.2</td>
<td>.996</td>
<td>1.000</td>
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<tr>
<td>.3</td>
<td>.799</td>
<td>.804</td>
</tr>
<tr>
<td>.4</td>
<td>.644</td>
<td>.651</td>
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<tr>
<td>.5</td>
<td>.519</td>
<td>.524</td>
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<td>.6</td>
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<td>.412</td>
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<tr>
<td>.7</td>
<td>.304</td>
<td>.305</td>
</tr>
<tr>
<td>.8</td>
<td>.201</td>
<td>.201</td>
</tr>
<tr>
<td>.9</td>
<td>.100</td>
<td>.100</td>
</tr>
</tbody>
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Offer Vectors are iid

Prop. $V(r)$ is convex.

Table 2: $V(n)$ when $X_1, \ldots, X_n$ are iid uniform $(0, 1)$ and $c = .1$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$V(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.553</td>
</tr>
<tr>
<td>2</td>
<td>1.273</td>
</tr>
<tr>
<td>3</td>
<td>2.035</td>
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<tr>
<td>4</td>
<td>2.826</td>
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<tr>
<td>5</td>
<td>3.637</td>
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<td>6</td>
<td>4.463</td>
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<tr>
<td>7</td>
<td>5.302</td>
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<tr>
<td>8</td>
<td>6.150</td>
</tr>
<tr>
<td>9</td>
<td>7.007</td>
</tr>
<tr>
<td>10</td>
<td>7.871</td>
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</tbody>
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