

Wavelet Based PDE Valuation of Derivatives

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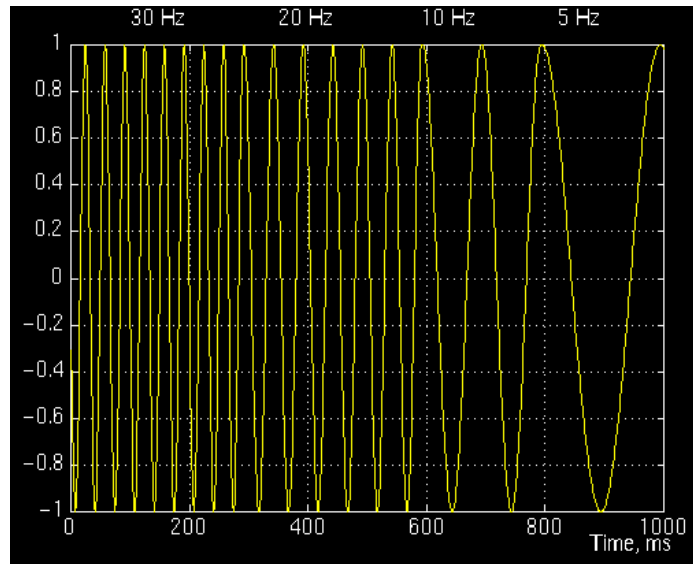
1. Introduction

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- What are wavelets ?
- **Wavelets** are nonlinear functions which can be **scaled** and **translated** to form a **basis** for the Hilbert space $L^2(\mathbb{R})$ of square integrable functions
- Wavelets generalize the **trigonometric functions** e^{ist} ($s \in \mathbb{R}$) which generate the classical **Fourier basis** for L^2
- Hence there are **wavelet** -- and fast **wavelet transforms** -- which generalize the time to frequency map of the Fourier transform to pick up **both** the **space and time** behavior of a function



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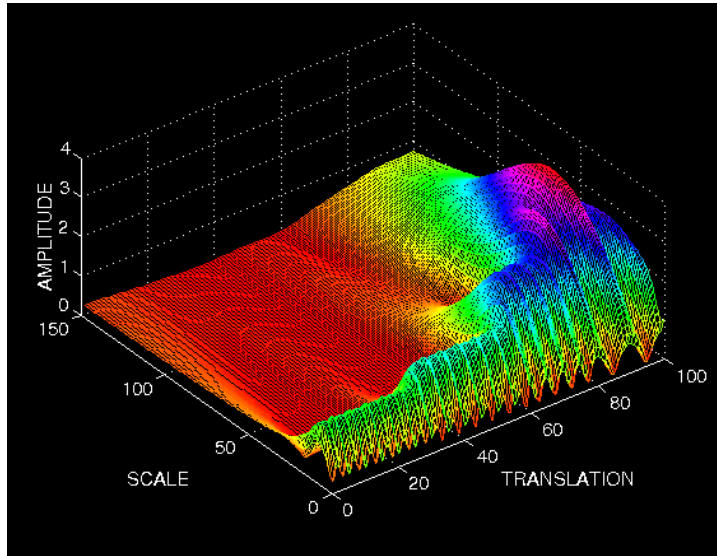


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Applications of wavelets

- Digital image compression
- Signal processing
- De-noising of signals --filtering
- Numerical analysis



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Wavelets and PDEs

A wavelet based approach to the solution of PDEs has been studied by

- Beylkin (1993)
- Vasilyev, Yuen, Pao (1997)
- Monasse and Perrier (1998)
- Prosser and Cant (1999, 2000)
- Dahmen *et al* (1999)
- Cohen *et al* (2000)



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Advantages of using wavelets to solve derivative valuation PDEs

- Wavelet PDE methods **combine** the advantages of both **spectral** (Fourier) and **finite-difference methods** and allow both **space and time dependent coefficients**
- Large classes of operators and functions are **sparse** or **sparse** to **high accuracy** when transformed into the wavelet domain
- Wavelets are suitable for problems with the **multiple spatial scales** common in financial problems
- Wavelets can be used for **nonlinear** terms
- Wavelets accurately represent the PDE solution in regions of **sharp transitions**



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2. Wavelet Transforms

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Scaling Functions and Wavelets

- The scaling function ϕ is the solution of a **dilation equation**

$$\phi(x) = \sqrt{2} \sum_{k=0}^{\infty} h_k \phi(2x - k)$$

where ϕ is normalised i.e. $\int_{-\infty}^{\infty} \phi(x) dx = 1$

- The **wavelet** ψ is defined in terms of the scaling function as

$$\psi(x) = \sqrt{2} \sum_{k=0}^{\infty} g_k \phi(2x - k)$$



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- The **basis functions** $\phi_{j,k}$ and $\psi_{j,k}$ are generated from ϕ and ψ through **scaling** and **translation** as

$$\phi_{j,k}(x) := 2^{-j/2} \phi(2^{-j}x - k) = 2^{-j/2} \phi\left(\frac{x - 2^j k}{2^j}\right)$$

$$\psi_{j,k}(x) := 2^{-j/2} \psi(2^{-j}x - k) = 2^{-j/2} \psi\left(\frac{x - 2^j k}{2^j}\right)$$



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- $h := \{h_k\}_{k=0}^{\infty}$ and $g := \{g_k\}_{k=0}^{\infty}$ are chosen so that dilations and translations of the wavelet ψ form an **orthonormal** basis of $L^2(\mathbb{R})$ i.e.

$$\int_{-\infty}^{\infty} \psi_{j,k}(x) \psi_{j',k'}(x) dx = \delta_{j,j'} \delta_{k,k'}$$

where δ denotes the Kronecker delta

- The two sequences of coefficients h and g are known from the signal processing literature as *filters*



- For any function $f \in L^2(\mathbb{R})$ there exists a matrix $\{d_{jk}\}$ such that

$$f(x) = \sum_{j \in \mathbb{Z}_+} \sum_{k \in \mathbb{Z}_+} d_{jk} \psi_{j,k}(x)$$

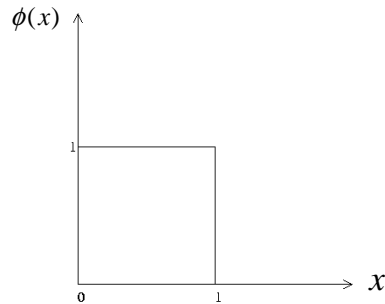
where

$$d_{jk} = \int_{-\infty}^{\infty} f(x) \psi_{j,k}(x) dx$$



- For example the **Haar wavelet** is defined by $h_0 = h_1 = 1/\sqrt{2}$
- For the Haar filter the solution to the dilation equation gives the unit **box function** for ϕ i.e.

$$\phi(x) = 1 \text{ for } x \in [0,1] \text{ and } 0 \text{ otherwise}$$



- The spaces spanned by $\phi_{j,k}$ and $\psi_{j,k}$ over the location parameter k with the scale parameter j fixed are usually denoted by

$$V_j = \text{span}_{k \in \mathbb{Z}_+} \phi_{j,k}$$

$$W_j = \text{span}_{k \in \mathbb{Z}_+} \psi_{j,k}$$

- To implement wavelet analysis on a computer we must have a **smallest** scale and a **largest** scale and thus we limit the range of the scale and location parameters j and k
- The spaces V_j and W_j are termed **approximation subspaces** and **scaling function subspaces** respectively



- These parameter ranges are chosen according to a required grid size in numerical computation
- The **orthogonal wavelet approximation** to a continuous function f is given by

$$f(x) \approx \sum_k d_{0,k} \psi_{0,k}(x) + \dots + \sum_k d_{J-1,k} \psi_{J-1,k}(x) + \sum_k s_{0,k} \phi_{0,k}(x)$$

where J is the **number** of resolution **scales** and k ranges from 0 to the **number of coefficients** in the specified component



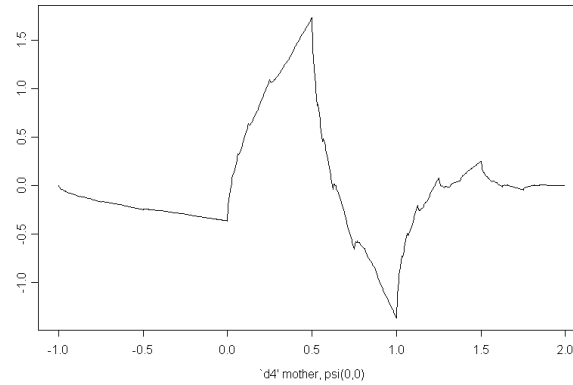
- For **Daubechies wavelets** the length L of the filters h and g is related to **exact** $(M-1)^{th}$ degree polynomial approximation for a fixed number J of resolution scales
- This is guaranteed by requiring the wavelet ψ to have $M := L/2$ vanishing moments i.e.

$$\int_{-\infty}^{\infty} \psi(x) x^m dx = 0$$

for $m=0, \dots, M-1$



Daubechies wavelet ψ



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Biorthogonal wavelets

Cohen, Daubechies and Feauveau (1992)

- **Four** basic function types -- two **primals** and two **duals**
- The biorthogonal wavelet approximation is expressed in terms of the dual wavelet functions

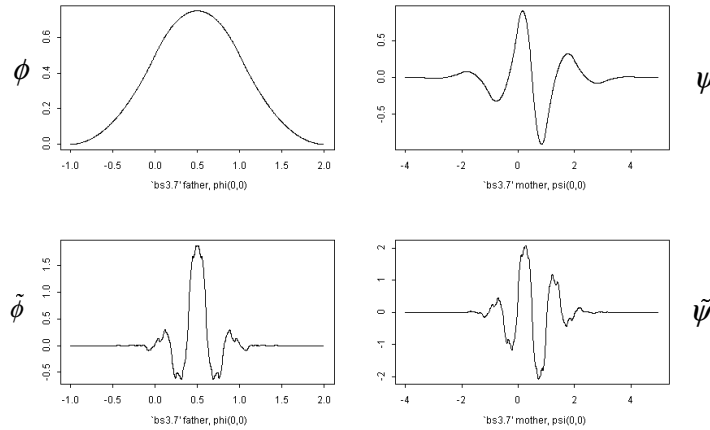
$$f(x) \approx \sum_k d_{0,k} \tilde{\psi}_{0,k}(x) + \dots + \sum_k d_{J-1,k} \tilde{\psi}_{J,k}(x) + \sum_k s_{0,k} \tilde{\phi}_{0,k}(x)$$



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Biorthogonal B-spline wavelets



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- Biorthogonal wavelets are not orthogonal but they satisfy the **biorthogonality relationships**

$$\int \phi_{j,k}(x) \tilde{\phi}_{j',k'}(x) dx = \delta_{j,j'} \delta_{k,k'}$$

$$\int \phi_{j,k}(x) \tilde{\psi}_{j',k'}(x) dx = 0$$

$$\int \psi_{j,k}(x) \tilde{\phi}_{j',k'}(x) dx = 0$$

$$\int \psi_{j,k}(x) \tilde{\psi}_{j',k'}(x) dx = \delta_{j,j'} \delta_{k,k'}$$

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Biorthogonal approach

- Biorthogonal wavelets are symmetrical and generalize the orthogonal wavelet approximation
- **Basis functions** for **biorthogonal wavelet spaces** are generated from the **primal scaling function** ϕ and the **dual scaling function** $\tilde{\phi}$
- Biorthogonal systems are thus derived from a paired hierarchy of **approximation subspaces**

$$\begin{aligned} \cdots V_{j-1} &\subset V_j \subset V_{j+1} \cdots \\ \cdots \tilde{V}_{j-1} &\subset \tilde{V}_j \subset \tilde{V}_{j+1} \cdots \end{aligned}$$



- **Wavelet innovation spaces** W_j and \tilde{W}_j are defined by

$$\begin{aligned} V_{j+1} &= V_j \oplus W_j \\ \tilde{V}_{j+1} &= \tilde{V}_j \oplus \tilde{W}_j \end{aligned}$$

with $\tilde{V}_j \perp W_j$ $V_j \perp \tilde{W}_j$

- **Basis functions** for the wavelet innovation spaces are generated by the primal and dual wavelets ψ and $\tilde{\psi}$
- Projections of a function f onto finite dimensional scaling function space V_j or wavelet space W_j are given by

$$\begin{aligned} P_{V_j} f(x) &= \sum_k \langle f(u), \tilde{\phi}_{j,k}(x) \rangle \phi_{j,k}(x) \\ P_{W_j} f(x) &= \sum_k \langle f(u), \tilde{\psi}_{j,k}(x) \rangle \psi_{j,k}(x) \end{aligned}$$



Biorthogonal interpolating wavelet transforms

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Deslauriers and Dubuc (1989) Donoho (1992)

- The **biorthogonal interpolating wavelet transform** has basis functions of the form

$$\phi_{j,k}(x) = \phi(2^j x - k)$$

$$\psi_{j,k}(x) = \phi(2^{j+1} x - 2k - 1)$$

$$\tilde{\phi}_{j,k}(x) = \delta(x - x_{j,k})$$

where δ is the Dirac delta function and $x_{j,k}$ is a grid point in the spatial dimension at scale level j

- An explicit form for $\tilde{\psi}$ is unknown but the corresponding wavelet coefficients can be derived using the biorthogonality conditions
- These wavelets are **interpolating** because the primal scaling function $\phi(x)$ is equal to 1 for $x = 0$ and 0 for $x \neq 0$ leading to polynomial interpolation of up to order $M-1$ between grid points



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Fast interpolating wavelet transform algorithm

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- The projection of a function f onto a finite dimensional scaling function space V_j is given by

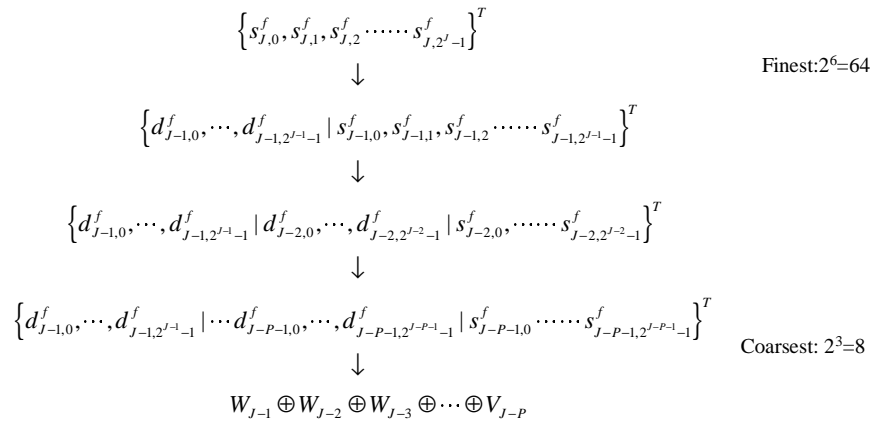
$$\begin{aligned} P_{V_j} f(x) &= \sum_k \langle f(u), \tilde{\phi}_{j,k}(x) \rangle \phi_{j,k}(x) \\ &= \sum_k f(k/2^j) \phi_{j,k}(x) \\ &:= \sum_k s_{j,k}^f \phi_{j,k}(x) \end{aligned}$$



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Fast interpolating wavelet transform algorithm structure



Example: $J := 6$ $P := 3$



Interpolating biorthogonal wavelet transform complexity

- The number of operations required for the transform algorithm for P resolution levels is $2M \sum_{i=I-P}^{I-1} 2^i = 2^{J-P} M (2^{P+1} - 1)$ as $2M$ filter coefficients define the primal scaling function which spans the space of polynomial of degree less than $M-I$
- Calculation of the wavelet coefficients $d_{j,k}^f$ for a given resolution level j can be accomplished in $2(M-I)+I$ floating point operations and the sub-sampling process for the scaling function coefficients requires a further 2^j operations for a total of $2^{j+1}M$ operations required per resolution level j
- For fixed J and P the fast interpolating wavelet transform algorithm is $O(M)$ for exact $(M-I)^{st}$ order polynomial approximation
- Since the finest resolution in a spatial grid of N points is $J = \log_2 N$ for fixed M and P the complexity of the transform is $O(N)$



3. Wavelets and PDEs

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Decomposition of differential operators

- Define $\partial_J^{(n)}$ such that

$$\partial_J^{(n)} f(x) := P_{V_J} \frac{d^n}{dx^n} P_{V_J} f(x)$$

- Repeated application of the approximation subspace decomposition gives

$$\partial_J^{(n)} f(x) := \left(P_{V_{J-P}} + \sum_{i=J-P}^{J-1} P_{W_i} \right) \frac{d^n}{dx^n} \left(P_{V_{J-P}} + \sum_{i=J-P}^{J-1} P_{W_i} \right) f(x)$$



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- For example the decomposition of the **first derivative operator** $\frac{d}{dx}$ is given by

$$\partial_J = \left(P_{V_{J-P}} + \sum_{i=J-P}^{J-1} P_{W_i} \right) \frac{d}{dx} \left(P_{V_{J-P}} + \sum_{i=J-P}^{J-1} P_{W_i} \right)$$

- Alternatively $\partial_J = W \partial_J^1 W^{-1}$ where $\partial_J^1 = P_{V_J} \frac{d}{dx} P_{V_J}$ and W and W^{-1} are orthogonal matrices denoting the **forward** and **inverse transforms** with $W = W' = W^{-1}$



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- Using the sampling nature of the dual scaling function ∂_J^1 can be written as to take account of domain boundaries as

$$\partial_J^1 = \begin{cases} 2^J \sum_{\alpha,k} s_{J,k}^f \frac{d\phi^L}{dx}(\alpha-k) \phi_{J,\alpha}^L & k = 0, \dots, M-1 \\ 2^J \sum_{\alpha,k} s_{J,k}^f \frac{d\phi}{dx}(\alpha-k) \phi_{J,\alpha} & k = M, \dots, 2^J - M \\ 2^J \sum_{\alpha,k} s_{J,k}^f \frac{d\phi^R}{dx}(\alpha-k) \phi_{J,\alpha}^R & k = 2^J - M + 1, \dots, 2^J \end{cases}$$



- The entire operator ∂_J^1 can thus be determined provided the values of $r_{\alpha-k}^{(1)} = d\phi(\alpha-k)/dx$ can be obtained
- An approach to determining filter coefficients $r_k^{(n)}$ for a derivative of order n is given by [Prosser and Cant \(1999\)](#)
- Their development provides analytic expressions for the resulting scaling function ϕ and its derivative filter coefficients



PDEs in 1 space dimension

- Consider a **first order nonlinear hyperbolic PDE** defined over an interval $\Omega = [x_l, x_r]$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} + S^{u/\rho} \quad x \notin \partial\Omega$$

$$\frac{\partial u}{\partial t} = -x^L(t) \quad x = x_l$$

$$\frac{\partial u}{\partial t} = -x^R(t) \quad x = x_r$$



- Its **wavelet transformed counterpart** is

$$\frac{\partial}{\partial t} \wp_{J-P}^{J-1}(u) = -\partial_J^{(1)} u + \wp_{J-P}^{J-1} S^{u/\rho} \quad x \notin \partial\Omega$$

where $\wp_{J-P}^{J-1} = \left(P_{V_{J-P}} + \sum_{i=J-P}^{J-1} P_{W_i} \right)$ and $\partial_J^{(1)}$ is the standard decomposition of $\frac{d}{dx}$ defined as $\wp_{J-P}^{J-1} \frac{d}{dx} \wp_{J-P}^{J-1}$

- Using the multiresolution strategy to discretize the problem we represent the domain $P+1$ times for a number P of different resolutions of the discretization
- There are P **wavelet spaces** and the coarse resolution **scaling function space** V_{J-P} ($P \geq 1$)
- In the transform domain each representation of the solution defined at some resolution P must be supplemented by **boundary conditions**



Wavelet method of lines

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- A traditional **finite difference scheme** replaces partial derivatives with algebraic approximations at grid points and solves the system of algebraic equations to obtain a numerical solution of the PDE
- The **method of lines** transforms the PDE into a vector ODE by replacing the spatial partial derivatives with their wavelet approximations but retaining the time derivatives
- The vector ODE is solved in time using a **stiff ODE solver**
- A method based on the **backward differentiation formula** (LSODE) from Lawrence Livermore Laboratories has been implemented in C/Fortran 90 on an IBM RS6000/590
- The complexity of the method is $O(N\tau)$ for time discretization τ



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PDEs in d space dimensions

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- The entire multiresolution wavelet machinery presented so far can be extended to several space dimensions d by taking straight forward **Cartesian products** of appropriate approximation and scaling subspaces -- i.e. **tensor products** of appropriate wavelet bases -- to result in a fast wavelet transform of $O(N)$ for $N := n^d$ for spatial discretization n
- The imposition of boundary conditions for nonlinearly bounded domains is nontrivial but these are fortunately rare in PDE derivative valuation problems which usually are **Cauchy problems on a strip**



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4. European Option Valuation

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- The **Black Scholes PDE** is

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS \frac{\partial C}{\partial S} - rC = 0$$

where S is **stock price**, σ is **volatility** and r is the **risk free rate** of interest

- Transform to the **heat diffusion equation**

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \quad \text{for } -\infty < x < \infty, \tau > 0$$

using $S = Xe^x$ and $t = T - 2\tau/\sigma^2$

- Then $C(S, T) := \max(S - X, 0)$ becomes for $k := 2r/\sigma^2$

$$C(x, \tau) = e^{-\frac{1}{2}(k-1)x - \frac{1}{2}(k+1)^2 \tau} Xu(x, \tau)$$



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European vanilla call

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- For a vanilla European call option the boundary conditions are

$$C(0, t) = 0, \quad C(S, t) \sim S \text{ as } S \rightarrow \infty$$

- The boundary conditions for the transformed PDE are

$$u(x, \tau) = 0 \text{ as } x \rightarrow -\infty$$

$$u(x, \tau) \sim e^{\frac{1}{2}(k+1)x + \frac{1}{4}(k+1)^2 \tau} \text{ as } x \rightarrow \infty$$

$$u(x, 0) = \max \left\{ e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0 \right\}$$

- The heat equation solution can be transformed back to original variables as

$$C(S, t) = X^{\frac{1}{2}(k+1)} S^{\frac{1}{2}(1-k)} e^{\frac{1}{8}(k+1)^2 \sigma^2 (T-t)} u \left(\log(S/X), \frac{1}{2} \sigma^2 (T-t) \right)$$

where $k = 2r/\sigma^2$



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Vanilla European Call

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- $S = 10$, strike = 10, $r = 5\%$, volatility = 20%, maturity = 1 Year
- Exact value: 1.04505

Wavelet method of lines

Space Steps	Time Steps	Value	Solution time (seconds)
64	60	1.03515	.05
128	100	1.04220	.10
256	200	1.04502	.13
512	200	1.04505	.30
1024	200	1.04505	.90

Crank-Nicolson Finite Difference Method

Space Steps	Time Steps	Value	Solution time (seconds)
64	60	1.03184	.02
128	100	1.04184	.04
256	200	1.04426	.09
512	200	1.04486	.16
1024	200	1.04501	.30
2000	200	1.04505	.57

- Speedup : 1.9



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Cash or nothing binary call

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- The **cash-or-nothing** call option has a **payoff**

$$\Pi(S) = BH(S - X)$$

where H is the **Heaviside function**, i.e. if at expiry the stock price $S > X$ the payoff is B
- The **boundary conditions** for this option in the **transformed domain** are

$$u(x, \tau) = 0 \text{ as } x \rightarrow -\infty$$

$$u(x, \tau) \sim \frac{B}{X} e^{\frac{1}{2}(k-1)x + \frac{1}{4}(k+1)^2 \tau} \text{ as } x \rightarrow \infty$$

$$u(x, 0) = e^{\frac{1}{2}(k-1)x} \frac{B}{X} H\left(e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0\right)$$



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Cash-or-nothing Call

- Same parameters as before, cash given $B=3$
- **Exact value 1.59297**

Wavelet method of lines

Space Steps	Time Steps	Value	Solution time (seconds)
128	100	1.49683	.10
256	200	1.54904	.13
512	200	1.59216	.30
1024	400	1.59288	1.02

Crank-Nicolson Finite Difference Method

Space Steps	Time Steps	Value	Solution time (seconds)
128	200	1.46296	.04
256	400	1.53061	.10
512	400	1.56391	.18
1024	400	1.58046	.31
2048	800	1.58872	1.35
4096	800	1.59285	2.56

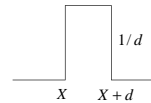
- **Speedup : 2.5**



Supershare binary call

- The **supershare binary call option** pays an amount $1/d$ if the stock price lies between X and $X+d$ at expiry
- The **payoff** of this option is

$$\Pi(S) = \frac{1}{d} (H(S - X) - H(S - X - d))$$



which becomes the Dirac delta function in the limit as $d \searrow 0$

- The **boundary conditions** for this option in the **transformed domain** are

$$u(x, \tau) = 0 \text{ as } x \rightarrow -\infty$$

$$u(x, \tau) \sim 0 \text{ as } x \rightarrow \infty$$

$$u(x, 0) = e^{\frac{1}{2}(k-1)x} (H(Xe^x - X) - H(Xe^x - X - d)) / dX$$



Supershare Binary Call

- Same parameters as before, parameter $d=3$
- **Exact value 0.13855**

Wavelet method of lines

Space Steps	Time Steps	Value	Solution time (seconds)
128	100	.12796	.10
256	200	.13310	.14
512	200	.13808	.30
1024	400	.13848	1.04

Crank-Nicolson Finite Difference Method

Space Steps	Time Steps	Value	Solution time (seconds)
128	200	.12369	.04
256	400	.13290	.09
512	400	.13435	.16
1024	400	.13666	.34
2048	800	.13787	1.35
4096	800	.13800	2.56
8000	800	.13835	5.11

- **Speedup : 4.9**



5. Cross-currency Swap Valuation

- We consider a 10 year **cross-currency cancellable fixed-fixed swap** with quarterly reset dates, a 2 day decision period and exchange of principal on first and last days
- Similar to the floating-floating deals discussed in [Dempster & Hutton \(1997\)](#) and [JP Morgan/Risk\(1999\)](#)
- 1-factor **extended Vasicek** yield curve models lead to a 3D parabolic PDE of the form

$$\frac{\partial V}{\partial t} - \frac{1}{2} \nabla [\Lambda(t)(\nabla V)'] = 0$$



- Solved for the normalized deal value 39 times in backwards time with a **backwards dynamic programming** reset of the initial condition at each quarter
- Boundary conditions are set at 3σ of the state distributions and the computed deal is **symmetric** --value 0-- with the same initial term structures, mean reversions and forward volatility specification in both economies
- A **3D wavelet method of lines** code in C/Fortran 90 and an **explicit finite difference** method in C have been implemented on an IBM RS6000/590



Cross Currency Swap

- Domestic Fixed Rate=10%, Foreign Fixed Rate=10%
- **Exact value : 0**

Wavelet method of lines

Discretization	Value	Solution time (seconds)
20 x 8 x 8 x 8	-0.00082	1.2
20 x 16 x 16 x 16	-0.00052	6.54
20 x 32 x 32 x 32	-0.00047	40.40
40 x 64 x 64 x 64	-0.00034	410.10
100 x 128 x 128 x 128	-0.00028	4240.30
160 x 256 x 256 x 256	-0.00025	53348.10

Explicit Finite Difference Method

	Value	Solution time (seconds)
20 x 8 x 8 x 8	-0.00109	0.28
20 x 16 x 16 x 16	-0.00101	1.70
20 x 32 x 32 x 32	-0.00074	16.82
40 x 64 x 64 x 64	-0.00058	188.10
100 x 128 x 128 x 128	-0.00046	2421.6
160 x 256 x 256 x 256	-0.00038	33341.8

- **Speedup : 81+**



6. 3-Factor Interest Rate Swap Valuation

3D Gaussian Model

- The **short rate** is given by

$$r(t) = s(t) + X_1(t) + X_2(t) + X_3(t)$$

where $s(t)$ is a deterministic function

$$dX_1 = \mu_1 X_1 dt + \sigma_1 dW_1$$

$$dX_2 = \mu_2 X_2 dt + \sigma_2 dW_2$$

$$dX_3 = \mu_3 X_3 dt + \sigma_3 dW_3$$

and $E[dW_i dW_j] = \rho_{ij} dt$

- Bond prices** have a closed form in this model



- If V denotes the value of the **derivative security**, under the spot measure $\exp(-\int_0^t r(u) du) V(X_{1t}, X_{2t}, X_{3t}, t)$ is a **martingale**.

- Using Ito's lemma the PDE satisfied by the value function is

$$\begin{aligned} & \mu_1 X_1 \frac{\partial V}{\partial X_1} + \mu_2 X_2 \frac{\partial V}{\partial X_2} + \mu_3 X_3 \frac{\partial V}{\partial X_3} + \frac{1}{2} \sigma_1^2 \frac{\partial^2 V}{\partial X_1^2} + \frac{1}{2} \sigma_2^2 \frac{\partial^2 V}{\partial X_2^2} + \frac{1}{2} \sigma_3^2 \frac{\partial^2 V}{\partial X_3^2} \\ & + \rho_{12} \sigma_1 \sigma_2 \frac{\partial^2 V}{\partial X_1 \partial X_2} + \rho_{13} \sigma_1 \sigma_3 \frac{\partial^2 V}{\partial X_1 \partial X_3} + \rho_{23} \sigma_2 \sigma_3 \frac{\partial^2 V}{\partial X_2 \partial X_3} - (s(t) + X_1 + X_2 + X_3)V + \frac{\partial V}{\partial t} = 0 \end{aligned}$$

- This PDE must be solved with appropriate **boundary conditions**



Deal Specification

Fixed for floating LIBOR swap

- **Payment** at end of each period is $p_j = Z\delta_j[L(t_{j-1}, t_j) - r^*]$ where r^* is the fixed **rate**, δ_j is the **accrual factor**, Z is the **notional principal** and $L(t_{j-1}, t_j)$ is the **LIBOR rate** for period t_{j-1}, t_j
- Payment $B(t_{j-1}, t_j)p_j$ at the end of each period
- Value of the **swap** at $t < T$ is the sum of the expected present values of all future payments
- PDE solved with **terminal condition** for each period given by $V(t_{j-1}^-) = B(t_{j-1}, t_j)p_j + V(t_{j-1})$ over the spatial grid points



Bermudan Swaption

- The **counterparty** has the option to enter the swap at the beginning of every period
- The **terminal condition** for each period is given by

$$V(t_j^-) = \max\{\Sigma, V(t_j)\}$$

where Σ represents the sum of the conditional expected present values of all future payments after t_j

- The PDE is solved for each period with the terminal condition given above



Bermudan Swaption

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- 1 Year, 2 Periods
- Fixed Rate=5%, Initial Flat Term Structure=5%
- MC value : 0.09111

Wavelet method of lines

Discretization	Value	Solution time (seconds)
40 x 16 x 16 x 16		
40 x 32 x 32 x 32		
80 x 64 x 64 x 64		
150 x 128 x 128 x 128		
280 x 256 x 256 x 128		

Dufort Frankel Explicit Finite Difference Method

Discretization	Value	Solution time (seconds)
40 x 16 x 16 x 16	.12005	.04
40 x 32 x 32 x 32	.10677	.65
80 x 64 x 64 x 64	.09814	20.27
150 x 128 x 128 x 128	.09328	461
300 x 256 x 256 x 128	.09156	4800

- Speedup :



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6. Conclusions and Further Work

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- $O(N)$ wavelet based PDE methods generalize $O(N \log_2 N)$ spectral methods **without their drawbacks**
- Wavelets are ideally suited for complex derivative valuations which involve several space scales e.g. due to **payoff curvature** or **discontinuities**
- Wavelets result in **greater accuracy** at a given discretization level with substantial speedups -- using prototype code -- over optimized finite difference codes in dimensions up to 3
- We are currently implementing a **thresholded** 3D wavelet code which should improve both speedup and memory use by an order of magnitude through **sparse wavelet representation**
- Then on to 2 and 3 factor cross currency swap valuation in 5 and 7D...



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