Pricing Options on Scalar Diffusions: An Eigenfunction Expansion Approach

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Abstract

This paper develops an eigenfunction expansion approach to pricing options on scalar diffusion processes. All derivative securities are unbundled into portfolios of primitive securities termed *eigensecurities*. Eigensecurities are eigenvectors of the pricing operator (present value operator). Pricing is then immediate by the linearity property of the pricing operator and the eigenvector property of eigensecurities. To illustrate the computational power of the method, we develop two applications: pricing vanilla, single- and double-barrier options under the constant elasticity of variance (CEV) process and interest rate knock-out options in the Cox-Ingersoll-Ross (CIR) term-structure model.

Keywords: option pricing, state-price density, eigensecurities, diffusion processes, eigenfunction expansion, Sturm-Liouville problem, CEV process, CIR process, barrier options

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1 Introduction

In this paper we develop an option pricing methodology based on unbundling derivative securities into portfolios of primitive securities termed *eigensecurities*. Eigensecurities are eigenvectors of the pricing operator (present value operator).

Arrow-Debreu securities, each paying one dollar in one specific state of nature and nothing in any other state, are the fundamental building blocks in asset pricing theory (see Duffie (1996)). In a continuum of states, the prices of Arrow-Debreu securities are defined by the state-price density, which gives for each state x the price of a security paying one dollar if the state falls between x and $x + dx$. If we know the functional form of the state-price density, we can price any European-style contingent claim by integrating the terminal payoff with the state-price density. In the diffusion setting, the state-price density can be found as a fundamental solution of the pricing partial differential equation (PDE) subject to some boundary conditions. Unfortunately, the task of solving the pricing PDE in closed form is often formidable, and no explicit analytical expressions for the state-price density are available in many cases of interest in applications.

In this paper we develop an alternative valuation methodology. Instead of using Arrow-Debreu securities to span the space of European-style contingent claims written on a scalar diffusion process, we introduce a concept of *eigensecurities* or eigenvectors of the pricing operator, as fundamental building blocks in our approach.[1] Eigensecurities diagonalize the pricing operator. All other European-style contingent claims with square-integrable payoffs are represented as portfolios of eigensecurities. Furthermore, the connection between eigensecurities and Arrow-Debreu securities can be established as follows. Arrow-Debreu securities themselves can be formally unbundled into portfolios of eigensecurities. This produces an eigenfunction expansion of the state-price density termed the *spectral resolution of the state-price density*. Depending on the nature of the diffusion process and boundary conditions, the spectrum can be discrete, continuous or mixed.[2]

In this paper we show that the eigenfunction expansion method is a powerful computational tool for derivatives pricing. Firstly, while the state-price density solves the boundary-value problem for the pricing *partial* differential equation (PDE), the eigensecurities are solutions to the *static* pricing equation without the time derivative term. In the scalar diffusion context, this static pricing equation can be interpreted as a second-order *ordinary* differential equation (ODE) of the Sturm-Liouville type.[3] Secondly, in cases where the state space is a finite interval with unmixed boundary conditions (e.g., absorbing or reflecting), the spectrum of the associated Sturm-Liouville problem is guaranteed to

[1]The foundations of semigroup pricing theory in a general Markov context are developed by Duffie (1985), Duffie and Garman (1985), and Garman (1985). See Ethier and Kurtz (1986) for the semigroup approach to Markov processes. See Dunford and Schwartz (1963) for the spectral theory of linear operators in function spaces.

[2]See Ito and McKean (1974) pp.149-161, Karlin and Taylor (1981) pp.330-340, McKean (1956), Schoutens (2000), and Wong (1964) for applications of eigenfunction expansions (called *eigen-differential expansions* by Ito and McKean) to diffusion processes.

[3]See Dunford and Schwartz (1963), Stakgold (1998), and Zwillinger (1998) for the account of the Sturm-Liouville theory.

be simple, purely discrete, and bounded below. Accordingly, under these boundary conditions eigenfunction expansions for security prices are infinite series. Moreover, eigenvalues ρ_n, $n = 1, 2, ...$, grow as n^2 and eigenfunction expansions converge rapidly, with contributions from the higher eigenfunctions suppressed by the factors $e^{-\rho_n T}$ (where T is time to maturity). Only a limited number of terms in the expansion are typically needed to achieve high accuracy in applications.

Several applications of the spectral method to problems in financial economics have already been considered in the literature. Hansen, Scheinkman, and Touzi (1998) develop spectral methods for econometric applications (estimation of scalar diffusions). Goldstein and Keirstead (1997) apply the eigenfunction expansion approach to the pricing of bonds under the short-rate processes with reflecting and absorbing boundaries. In an interesting recent paper, Lewis (1998) applies the eigenfunction expansion approach to solve two problems in continuous-time finance: pricing options on stocks that pay dividends at a constant dollar rate and pricing bonds under a short-rate process with non-linear drift. Lewis (2000) applies the eigenfunction expansion approach to the analysis of stochastic volatility models. Madan and Milne (1994) approximate contingent claim prices by series of Hermite polynomials.

In this paper we develop the general eigenfunction expansion method for claims contingent on scalar diffusions and develop two specific applications: pricing vanilla, single- and double-barrier options under Cox's constant elasticity of variance (CEV) process and interest rate knock-out options in the Cox-Ingersoll-Ross (CIR) term-structure model.

The remainder of the paper is organized as follows. In Section 2 we consider a motivating example of a double-barrier option under the lognormal process and develop the eigenfunction expansion for this case. In Section 3 we formally introduce eigensecurities and develop the general methodology of pricing options on scalar diffusions via eigenfunction expansions. In Section 4 we apply the method to the case of vanilla, single- and double-barrier options under the CEV process. Our main result is the analytical inversion of the Laplace transforms in maturity for CEV barrier option prices obtained by Davydov and Linetsky (2000). In Section 5 we apply the method to interest rate knock-out options in the CIR term-structure model. Section 6 concludes the paper. Proofs are collected in the Appendix.

2 A Motivating Example

Consider a double-barrier call option with the strike price K, expiration date T, and two knock-out barriers L and U, $0 < L < K < U$ (see Geman and Yor (1996), Kunitomo and Ikeda (1992), Pelsser (2000), Schroder (1999), Zhang (1997)). The knock-out provision renders the option worthless as soon as the underlying price leaves the price range (L, U). In this Section we assume that under the risk-neutral measure Q the underlying asset price follows a geometric Brownian motion

$$S_t = Se^{\sigma(B_t+\nu t)}, \quad t \geq 0, \tag{1}$$

where $\{B_t, t \geq 0\}$ is a standard Brownian motion starting at the origin at $t = 0$, σ is the constant volatility, r is the constant risk-free interest rate, q is the constant dividend yield, S is the initial asset price at $t = 0$, and

$$\nu = \frac{1}{\sigma}\left(r - q - \frac{\sigma^2}{2}\right). \tag{2}$$

The double barrier call payoff is

$$\mathbf{1}_{\{\mathcal{T}_{(L,U)} > T\}}(S_T - K)^+, \tag{3}$$

where $\mathcal{T}_{(L,U)}$ is the first exit time from the range (L, U), $\mathcal{T}_{(L,U)} = \inf\{t : S_t \notin (L, U)\}$, $\mathbf{1}_{\{A\}}$ is the indicator function of the event A, and $x^+ \equiv \max(x, 0)$. Then the double-barrier call price at $t = 0$ is given by the discounted risk-neutral expectation of the payoff

$$e^{-rT} E_S[\mathbf{1}_{\{\mathcal{T}_{(L,U)} > T\}}(S_T - K)^+], \tag{4}$$

where the subscript in E_S signifies that the expectation is taken with respect to the process (1) starting at $S_0 = S$ at time $t = 0$.

Consider a more general problem of pricing a European-style, double-barrier contingent claim with a square-integrable payoff $f \in \mathcal{L}_2[L, U]$:

$$V_f(S, T) = e^{-rT} E_S[\mathbf{1}_{\{\mathcal{T}_{(L,U)} > T\}} f(S_T)]. \tag{5}$$

Proposition 1 *Let $\mathfrak{m}(X)$ be the speed density of the geometric Brownian motion (1)*

$$\mathfrak{m}(X) = \frac{2}{\sigma^2} X^{\frac{2\nu}{\sigma} - 1}. \tag{6}$$

Let $\mathcal{H} = \mathcal{L}_2([L, U], \mathfrak{m})$ be the Hilbert space of functions on the interval $[L, U]$ square-integrable with the weight $\mathfrak{m}$ and endowed with the inner product

$$(f, g) = \int_L^U f(X) g(X) \mathfrak{m}(X) dX, \tag{7}$$

and $\{\varphi_n(X), n = 1, 2, ...\}$ — a set of functions in $\mathcal{H}$ defined by

$$\varphi_n(X) := \sqrt{\frac{\sigma}{u}} X^{-\frac{\nu}{\sigma}} \sin\left[\frac{\pi n}{\sigma u} \ln\left(\frac{X}{L}\right)\right], \quad n = 1, 2, ..., \quad x \in [L, U], \tag{8}$$

where

$$u := \frac{1}{\sigma} \ln\left(\frac{U}{L}\right). \tag{9}$$

(i) Functions φ_n are eigenvectors *(*eigenfunctions*) of the pricing operator for the problem with two absorbing barriers:*

$$e^{-rT} E_S[\mathbf{1}_{\{\mathcal{T}_{(L,U)} > T\}} \varphi_n(S_T)] = e^{-(r+\rho_n)T} \varphi_n(S), \tag{10}$$

where

$$\rho_n = \frac{\nu^2}{2} + \frac{n^2\pi^2}{2u^2}. \tag{11}$$

(ii) Functions φ_n form a complete, orthonormal basis in $\mathcal{H}$,

$$(\varphi_n, \varphi_n) = 1, \quad (\varphi_n, \varphi_m) = 0, \quad n \neq m. \tag{12}$$

Any payoff $f \in \mathcal{H}$ is in their span:

$$f = \sum_{n=1}^{\infty} f_n \, \varphi_n, \tag{13}$$

where

$$f_n = (f, \varphi_n) \tag{14}$$

and convergence is in the norm of the Hilbert space.

(iii) The price of the double-barrier claim (5) is:

$$V(S,T) = \sum_{n=1}^{\infty} e^{-(r+\rho_n)T} f_n \, \varphi_n(S). \tag{15}$$

Proposition 1 unbundles any European-style, double-barrier claim with the square-integrable payoff f into a portfolio of primitive double-barrier securities with the terminal payoffs φ_n (*eigensecurities* with the *eigenpayoffs* φ_n). Eigenpayoffs form a complete orthonormal basis in the space of all $\mathcal{L}_2$ payoffs. For the double-barrier call option with the payoff (3) we have in particular:

Corollary 1 *The price of the double-barrier call (4) is given by the eigenfunction expansion of the form (15) with the coefficients*

$$f_n = \frac{L^{\frac{\nu}{\sigma}}}{\sqrt{\sigma u}} [L \, \psi_n(\nu + \sigma) - K \, \psi_n(\nu)], \tag{16}$$

$$\psi_n(a) := \frac{2}{\omega_n^2 + a^2} \left[e^{ak} (\omega_n \cos(\omega_n k) - a \sin(\omega_n k)) - (-1)^n \omega_n e^{au} \right], \tag{17}$$

$$\omega_n := \frac{n\pi}{u}, \quad k := \frac{1}{\sigma} \ln \left(\frac{K}{L} \right). \tag{18}$$

The observation of practical interest is that ρ_n grow as n^2 as n increases, and contributions from the higher eigenfunctions are suppressed by the factors $e^{-\rho_n T}$. As a result, the eigenfunction expansion (15) converges so rapidly that only the first several terms are needed to achieve high accuracy in option pricing applications with typical parameter values. Table 1 (page 15) illustrates convergence of the eigenfunction expansion for double-barrier calls with one and twelve months to expiration and $S = K = 100$, $L = 90$, $U = 120$, $\sigma = 0.25$, $r = 0.1$, $q = 0$ (the case $\beta = 0$ corresponds to the lognormal process). For the twelve month option, the first one or two terms in the eigenfunction expansion are sufficient to achieve the accuracy of 10^{-4}. As maturity decreases, more terms in the series are required to achieve the same accuracy. For the one month option, the first five terms are needed.

To conclude this Section, we note that the price of the double-barrier option vanishes in the limit $T \to \infty$. Analytically, this follows from the fact that the first eigenvalue is strictly positive, $\rho_1 > 0$. Probabilistically, this follows from the fact that the stock price eventually hits a barrier with probability one.

3 Spectral Methods for Options on Scalar Diffusions

In this Section we take an equivalent martingale measure Q as given and assume that under Q the state variable in our economy follows a one-dimensional, time-homogeneous diffusion process $\{X_t, t \geq 0\}$ taking values in some interval $D \subset \mathbf{R}$ with the end-points $\mathfrak{l}$ and $\mathfrak{r}$, $-\infty \leq \mathfrak{l} < \mathfrak{r} \leq \infty$, and solving the stochastic differential equation

$$dX_t = b(X_t)dt + a(X_t)dB_t, \quad t > 0, \quad X_0 = x, \quad x \in (\mathfrak{l}, \mathfrak{r}), \tag{19}$$

where $\{B_t, t \geq 0\}$ is a standard Brownian motion, and functions $a(x)$ and $b(x)$ are diffusion and drift coefficients, respectively. We assume that the functions $a(x)$ and $b(x)$ are continuous and $a(x) > 0$ for all $x \in (\mathfrak{l}, \mathfrak{r})$. The boundary behavior at the end-points $\mathfrak{l}$ and $\mathfrak{r}$ depends on the behavior of functions $a(x)$ and $b(x)$ as $x \to \mathfrak{l}$ and $x \to \mathfrak{r}$ (the boundary characterization of one-dimensional diffusions due to Feller is described in Chapter 15 of Karlin and Taylor (1981) and Chapter 2 of Borodin and Salminen (1996)). If any of the end-points is a regular boundary, we adjoin a killing boundary condition at that end-point. We also assume that the instantaneous risk-free interest rate is a function of the state variable, $r_t = R(X_t)$, and $R(x)$ is non-negative and continuous for all $x \in (\mathfrak{l}, \mathfrak{r})$.

Let $I = [L, U]$ be an interval in the interior of D, $\mathfrak{l} < L < U < \mathfrak{r}$, and $x \in (L, U)$. Let f be a square-integrable function on I. Consider a double-barrier claim that pays off an amount $f(X_T)$ at expiration $T > 0$ if the process X does not leave the interval (L, U) prior to expiration, and zero otherwise. Then the price of this double-barrier claim at time $t = 0$ is given by

$$V(x, T) = E_x \left[e^{-\int_0^T R(X_t)dt} f(X_T) \mathbf{1}_{\{\mathcal{T}_{(L,U)} > T\}} \right], \tag{20}$$

where the subscript x in E_x signifies that the process X starts at x at $t = 0$, and $\mathcal{T}_{(L,U)} = \inf\{t : X_t \notin (L, U)\}$. The following proposition summarizes the situation.

Proposition 2 *Let $\mathfrak{s}$ and $\mathfrak{m}$ be the scale and speed densities*[4] *of the diffusion process (19):*

$$\mathfrak{s}(x) = \exp\left\{-\int^x \frac{2b(y)}{a^2(y)}dy\right\}, \tag{21}$$

$$\mathfrak{m}(x) = \frac{2}{a^2(x)\mathfrak{s}(x)}. \tag{22}$$

Let $\mathcal{H} = \mathcal{L}_2([L,U],\mathfrak{m})$ be the Hilbert space of functions on the interval $[L,U]$ square-integrable with the weight $\mathfrak{m}$ and endowed with the inner product

$$(f,g) = \int_L^U f(x)g(x)\mathfrak{m}(x)dx. \tag{23}$$

(i) $\mathcal{H}$ admits a complete orthonormal basis $\{\varphi_n(x), n = 1,2,...\}$ such that φ_n are eigenvectors (eigenfunctions) of the pricing operator

$$E_x\left[e^{-\int_0^T R(X_t)dt}\mathbf{1}_{\{\mathcal{T}_{(L,U)}>T\}}\varphi_n(X_T)\right] = e^{-\rho_n T}\varphi_n(x) \tag{24}$$

for some $0 < \rho_1 < \rho_2 < \ldots < \rho_n < \ldots$ with $\rho_n \to \infty$ as $n \to \infty$. Any payoff $f \in \mathcal{H}$ is in the span of eigenpayoffs *φ_n:*

$$f = \sum_{n=1}^{\infty} f_n\,\varphi_n, \quad f_n = (f,\varphi_n), \tag{25}$$

and convergence is in the norm of the Hilbert space.

(ii) Let $\mathcal{A}$ be the second-order differential operator[5]

$$\mathcal{A}u \equiv -\frac{1}{2}a^2(x)\frac{d^2u}{dx^2} - b(x)\frac{du}{dx} + R(x)u = -\frac{1}{\mathfrak{m}(x)}\frac{d}{dx}\left[\frac{1}{\mathfrak{s}(x)}\frac{du}{dx}\right] + R(x)u. \tag{26}$$

*The eigenvalue-eigenfunction pairs (ρ_n,φ_n) solve the second-order ODE subject to the absorbing boundary conditions (*regular Sturm-Liouville boundary-value problem[6]*):*

$$\mathcal{A}u = \rho u, \;\; x \in (L,U), \;\; u(L) = 0, \;\; u(U) = 0. \tag{27}$$

(iii) The price of the double-barrier claim (20) is:

$$V(x,T) = \sum_{n=1}^{\infty} e^{-\rho_n T} f_n\,\varphi_n(x). \tag{28}$$

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[4]See Karlin and Taylor (1981), p.194, Karatzas and Shreve (1991), p.343, and Borodin and Salminen (1996), p.17, for discussions of scale and speed densities. Our definition of the speed density coincides with that of Karatzas and Shreve (1991) and Borodin and Salminen (1996) and differs from Karlin and Taylor (1981) who do not include 2 in the definition.

[5]The second equality in (26) follows from the definitions of the scale and speed densities (21) and (22).

[6]See Dunford and Schwartz (1963), Pruess, Fulton and Xie (1996), Stakgold (1998) pp.435-490, and Zwillinger (1998) pp.94-99 for the account of the theory of Sturm-Liouville boundary-value problems.

Proposition 2 generalizes Proposition 1 to the case of arbitrary continuous functions $b(x)$, $a(x) > 0$, and $R(x) \geq 0$ (in the example of Section 2 we have $b(x) = (r-q)x$, $a(x) = \sigma x$ and $R(x) = r$). It unbundles any European-style, double-barrier contingent claim with the $\mathcal{L}_2$ payoff into a portfolio of *eigensecurities* with *eigenpayoffs* φ_n. The pricing is then automatic by the linearity property of the pricing operator and the eigenvector property of the eigenpayoffs (24). From the practical standpoint, all the work is at the stage of determining the eigenvalues ρ_n and the corresponding normalized eigenfunctions φ_n. This is accomplished by solving the regular Sturm-Liouville boundary value problem (26)-(27).

The observation of practical importance is that, as in the example of Section 2, the eigenvalues ρ_n of the regular Sturm-Liouville problem on the finite interval with two absorbing boundary conditions grow as n^2 as $n \to \infty$ (see Zwillinger (1998)), contributions from the higher eigenfunctions are suppressed by the factors $e^{-\rho_n T}$, and eigenfunction expansions for double-barrier contingent claim prices and hedge ratios converge rapidly for typical parameter values.

When $f(x) = \delta(x-y)$ for some fixed $y \in (L,U)$ ($\delta(x)$ is the Dirac delta-function), the continuous state-price density $p(t;x,y)$ with the two absorbing boundary conditions at L and U has a formal spectral representation (termed *spectral resolution of the state-price density*)

$$p(t;x,y)dy \equiv E_x\left[e^{-\int_0^t R(X_u)du}\mathbf{1}_{\{\mathcal{T}_{(L,U)}>t\}}; X_t \in dy\right] = \sum_{n=1}^{\infty} e^{-\rho_n t}\varphi_n(x)\varphi_n(y)\mathfrak{m}(y)dy. \quad (29)$$

For a complex λ with $\mathrm{Re}(\lambda) > 0$, introduce a *resolvent kernel* or *Green's function*

$$G_\lambda(x,y) := \int_0^\infty e^{-\lambda t}p(t;x,y)dt. \quad (30)$$

From Eq.(29), the resolvent kernel of the regular Sturm-Liouville problem with two absorbing boundary conditions can be represented as

$$G_\lambda(x,y) = \mathfrak{m}(y)\sum_{n=1}^{\infty}\frac{\varphi_n(x)\varphi_n(y)}{\lambda+\rho_n}. \quad (31)$$

Continuing the right-hand side of Eq.(31) to the whole λ plane, for each $x, y \in (L,U)$ the Green's function is a meromorphic function in the λ plane with simple poles at $\lambda = -\rho_n$, $n = 1, 2, \ldots$, and residues $\mathfrak{m}(y)\varphi_n(x)\varphi_n(y)$. In practice, one way to determine the eigenvalues and the corresponding normalized eigenfunctions φ_n of the regular Sturm-Liouville problem is to construct the Green's function in such a way that we can keep track of its dependence on λ, and then find its poles and calculate the residues.

We note that for each $y \in (L,U)$ the Green's function $G_\lambda(x,y)$ is a unique continuous solution of the inhomogeneous ODE with the two absorbing boundary conditions:[7]

$$(\mathcal{A}+\lambda)\,G_\lambda(x,y)=\delta(x-y),\ \ x\in(L,U),\ \ G_\lambda(L,y)=G_\lambda(U,y)=0. \tag{32}$$

The solution to this boundary-value problem can be constructed as follows (see Stakgold (1998), p.441). For each complex λ, let $\xi_\lambda(x)$ and $\eta_\lambda(x)$ be the unique solutions of the homogeneous ODE

$$(\mathcal{A}+\lambda)\,u=0,\ \ x\in(L,U) \tag{33}$$

with the initial conditions (prime denotes differentiation in x)

$$\xi_\lambda(L)=0,\ \ \xi_\lambda'(L)=1 \tag{34}$$

and

$$\eta_\lambda(U)=0,\ \ \eta_\lambda'(U)=-1. \tag{35}$$

For each x, the $\xi_\lambda(x)$ and $\eta_\lambda(x)$ are *entire* functions of λ (analytic in the whole λ plane).[8] By (33) and (26), the Wronskian of the functions $\eta_\lambda(x)$ and $\xi_\lambda(x)$ is of the form

$$W(\eta_\lambda(x),\xi_\lambda(x))\equiv\eta_\lambda(x)\xi_\lambda'(x)-\xi_\lambda(x)\eta_\lambda'(x)=C(\lambda)\mathfrak{s}(x), \tag{36}$$

where $\mathfrak{s}(x)$ is the scale density (21) and $C(\lambda)$ is independent of x but may depend on λ. Then the Green's function of the regular Sturm-Liouville problem with two absorbing boundary conditions can be taken in the form (Stakgold (1998), p.441) ($x\wedge y:=\min(x,y)$, $x\vee y:=\max(x,y)$):

$$G_\lambda(x,y)=\mathfrak{m}(y)\frac{\xi_\lambda(x\wedge y)\eta_\lambda(x\vee y)}{C(\lambda)}. \tag{37}$$

Since ξ and η are entire functions of λ, so are ξ', η', W, and $C(\lambda)$. Let $\lambda=-\rho$ be a zero of C, i.e. $C(-\rho)=0$. Then the Wronskian of $\xi_{-\rho}(x)$ and $\eta_{-\rho}(x)$ vanishes, and these functions are linearly dependent. In view of their initial values neither function can vanish identically in x. Therefore $\xi_{-\rho}(x)$ is a non-trivial constant multiple of $\eta_{-\rho}(x)$, and both functions satisfy the two boundary conditions and the ODE in (27). Thus, ρ is an eigenvalue of (27) with eigenfunction (not normalized) $\xi_{-\rho}(x)$. From (31) it is clear that at a negative of an eigenvalue $G_\lambda(x,y)$ has a simple pole, and therefore C must vanish. Thus, we conclude that the (simple) zeros of $C(\lambda)$ are located along the negative real axes and coincide with the negatives of eigenvalues of the Sturm-Liouville problem (27). We label the eigenvalues $0<\rho_1<\rho_2<\ldots<\rho_n<\ldots$ with $\rho_n\to\infty$ as $n\to\infty$, and

$$\xi_{-\rho_n}(x)=A_n\eta_{-\rho_n}(x), \tag{38}$$

[7] This follows from the fact that the state-price density solves the PDE $\mathcal{A}p=-\frac{\partial p}{\partial t}$, $x\in(L,U)$, $t\in(0,\infty)$, with the initial condition $p(0;x,y)=\delta(x-y)$, $x\in(L,U)$, and boundary conditions $p(t;L,y)=p(t;U,y)=0$, $t\in[0,\infty)$.

[8] This follows from the fact that $\xi_\lambda(x)$ and $\eta_\lambda(x)$ satisfy initial conditions independent of λ and an ODE where λ appears analytically (see Stakgold (1998), p.441).

where A_n is a real non-zero constant. Thus, $\xi_{-\rho_n}(x)$ (or $\eta_{-\rho_n}(x)$) is a real eigenfunction corresponding to the simple positive eigenvalue ρ_n. Neither $\xi_{-\rho_n}(x)$ nor $\eta_{-\rho_n}(x)$ is normalized. To find the normalized eigenfunctions, we note that the residue of $G_\lambda(x,y)$ at $\lambda = -\rho_n$ is

$$\mathfrak{m}(y)\frac{\xi_{-\rho_n}(x \wedge y)\eta_{-\rho_n}(x \vee y)}{C'(-\rho_n)} = \mathfrak{m}(y)\frac{\xi_{-\rho_n}(x)\xi_{-\rho_n}(y)}{A_n C'(-\rho_n)} = \mathfrak{m}(y)\frac{A_n\eta_{-\rho_n}(x)\eta_{-\rho_n}(y)}{C'(-\rho_n)}, \quad (39)$$

where

$$C'(-\rho_n) = \left[\frac{dC(\lambda)}{d\lambda}\right]\Bigg|_{\lambda=-\rho_n}. \quad (40)$$

On the other hand, from (31) the residue of $G_\lambda(x,y)$ at $\lambda = -\rho_n$ is equal to $\mathfrak{m}(y)\varphi_n(x)\varphi_n(y)$, and we recognize that the normalized eigenfunction $\varphi_n(x)$ is given by:

$$\varphi_n(x) = \pm\frac{\xi_{-\rho_n}(x)}{\sqrt{A_n C'(-\rho_n)}} = \pm\sqrt{\frac{A_n}{C'(-\rho_n)}}\eta_{-\rho_n}(x). \quad (41)$$

Thus, from the practical standpoint, the problem of finding eigenvalues and eigenfunctions reduces to solving the two initial value problems for the $\xi_\lambda(x)$ and $\eta_\lambda(x)$, calculating their Wronskian, and determining its zeros.

Remark 1. Continuous Dividend Streams. So far we have limited our discussion to cash flows that occur at some pre-specified future time $T > 0$. Our results can be straightforwardly extended to continuous dividend streams. Consider a security with dividends paid continuously during $[0,T]$. The dividends stop at time T or the first exit time $\mathcal{T}_{(L,U)}$, whichever comes first. Let $f_t = f(X_t)\mathbf{1}_{\{\mathcal{T}_{(L,U)}>t\}}$ be the dividend-rate process, so that the cumulative dividend process of a security is $D_t = \int_0^t f(X_u)\mathbf{1}_{\{\mathcal{T}_{(L,U)}>u\}}du$. Then the risk-neutral pricing formula is (Duffie (1996), p.116-8)

$$V(x,T) = E_x\left[\int_0^T e^{-\int_0^t R(X_u)du} f(X_t)\mathbf{1}_{\{\mathcal{T}_{(L,U)}>t\}}dt\right].$$

Application of Eq.(42) and Fubini's theorem yields the result for continuous dividend streams:

$$V(x,T) = \sum_{n=1}^{\infty}\left(\frac{1-e^{-\rho_n T}}{\rho_n}\right) f_n\varphi_n(x), \quad f_n = (f,\varphi_n).$$

Remark 2. Singular Problems and Continuous Spectra. In the preceding discussion we limited ourselves to double-barrier claims that knock out as soon as the underlying state variable leaves some pre-specified finite interval in the interior of the state space D. In this case the contingent claim pricing problem is reduced to the regular Sturm-Liouville problem with two absorbing boundary conditions at the endpoints of the interval. Consider now a contingent claim without knock-out barriers and a terminal payoff $f \in \mathcal{L}_2(D,\mathfrak{m})$. The pricing problem reduces to the Sturm-Liouville problem on the entire state space D with the end-points $\mathfrak{l}$ and $\mathfrak{r}$. If the interval D is finite, $\mathfrak{s}(x)$, $\mathfrak{m}(x)$, and $R(x)$ are continuous and $\mathfrak{m}(x) > 0$ and $\frac{1}{\mathfrak{s}(x)} > 0$ the open interval $(\mathfrak{l}, \mathfrak{r})$, and $\mathfrak{s}(x)$, $\mathfrak{m}(x)$, and $R(x)$ are absolutely integrable near both end-points $\mathfrak{l}$ and $\mathfrak{r}$, then the Sturm-Liouville problem is said to be *regular*. Otherwise, the problem is *singular*. For a regular problem with two absorbing boundary conditions the spectrum is simple, purely discrete, and bounded below, and Proposition 2 holds in the limiting case $L = \mathfrak{l}$ and $U = \mathfrak{r}$. In contrast, the spectrum of a singular problem can be discrete, continuous, or mixed, and further analysis is needed to determine the nature of the spectrum in each case. The complete classification scheme for singular Sturm-Liouville problems can be found in Pruess, Fulton and Xie (1996) and Zwillinger (1998) p.97. For options with a single upper (lower) knock-out barrier, the nature of the spectrum of the pricing problem will depend on the behavior of the functions $\mathfrak{s}(x)$, $\mathfrak{m}(x)$, and $R(x)$ at the left boundary $\mathfrak{l}$ (right boundary $\mathfrak{r}$), respectively. The formal spectral representation (*spectral resolution*) of the state-price density in the general case with mixed spectrum takes the form

$$p(t; x, y)dy = \left(\sum_{n \in \sigma_p} e^{-\rho_n t} \varphi_n(x) \varphi_n(y) + \int_{\rho \in \sigma_c} e^{-\rho t} \varphi_\rho(x) \varphi_\rho(y) d\rho \right) \mathfrak{m}(y) dy, \qquad (42)$$

where σ_p is the discrete (point) spectrum and σ_c is the continuous spectrum.

4 Barrier Options under the CEV Process

4.1 The CEV Process

In this Section we specialize to the *constant elasticity of variance* (CEV) process of Cox (1975). We assume that under the risk-neutral measure Q the asset price follows the CEV process[9]

$$dS_t = \mu S_t dt + \delta S_t^{\beta+1} dB_t, \ \ t > 0, \ \ S_0 = S > 0, \qquad (43)$$

where the risk-neutral drift rate is $\mu = r - q$ (r is the constant risk-free rate and q is the dividend yield). The CEV specification (43) nests the lognormal model of Black and Scholes (1973) and Merton (1973) ($\beta = 0$) and the absolute diffusion ($\beta = -1$) and square-root ($\beta = -1/2$) models of Cox and Ross (1976) as particular cases. For $\beta < 0$ ($\beta > 0$), the local volatility $\sigma(S) = \delta S^\beta$ is a decreasing (increasing) function of the asset price. The two model parameters β and δ can be interpreted as the *elasticity of the local volatility function*, $d\sigma/dS = \beta\sigma/S$, and the *scale parameter* fixing the initial instantaneous volatility at time $t = 0$, $\sigma_0 = \sigma(S_0) = \delta S_0^\beta$. Cox (1975) originally studied the case $\beta < 0$. Emanuel and MacBeth (1982) extended his analysis to the case $\beta > 0$. Cox originally

[9]Our notation is slightly different from Cox (1975). Our parameter β is defined as the elasticity of the local volatility function. Cox's parameter θ in $dS_t = \mu S_t dt + \delta S_t^{\theta/2} dB_t$ is defined as the elasticity of the instantaneous variance of the asset price. The two parameters are related by: $\beta + 1 = \theta/2$.

restricted the elasticity parameter to the range $-1 \leq \beta \leq 0$. However, Reiner (1994) and Jackwerth and Rubinstein (1998) find that typical values of the CEV elasticity implicit in the post-crash S&P 500 stock index option prices are as low as $\beta = -4$. They term the model with $\beta < -1$ *unrestricted CEV*.

According to Feller's classification of boundaries, for $\beta < 0$ infinity is a natural boundary for the CEV diffusion. For $-1/2 \leq \beta < 0$, the origin is an exit boundary. For $\beta < -1/2$, the origin is a regular boundary point, and is specified as a killing boundary by adjoining a killing boundary condition. For $\beta > 0$, the origin is a natural boundary and infinity is an entrance boundary (see Appendix B in Davydov and Linetsky (2000) for the treatment of $\beta > 0$ case). In this paper we will focus on the CEV process with $\beta < 0$ and $\mu > 0$ ($r > q$). This process is used to model the volatility (half)smile effect in the equity index options market.

The closed-form pricing formulas for vanilla calls and puts under the CEV process are derived by Cox (1975) (see also Schroder (1989) and Davydov and Linetsky (2000) and references therein). The problem of pricing single- and double-barrier options under the CEV process is examined by Boyle and Tian (1999) in the numerical trinomial lattice framework and by Davydov and Linetsky (2000) in the analytical framework. Davydov and Linetsky (2000) derive closed-form expressions for Laplace transforms of single- and double-barrier option prices in time to maturity. The Laplace transforms are then inverted numerically using the Euler numerical inversion algorithm of Abate and Whitt (1995) (see Fu, Madan and Wang (1997) and Davydov and Linetsky (2000b) for applications of the Euler inversion algorithm to option pricing problems). In this paper we develop eigenfunction expansions for single and double barrier option prices under the CEV process. These eigenfunction expansions invert the Laplace transforms of Davydov and Linetsky (2000) in *closed form*.

Remark 3. CEV, Feller, and Radial Ornstein-Uhlenbeck Diffusions. The CEV process is related to several diffusions prominent in the stochastic processes literature. Let $\{S_t, t \geq 0\}$ be the CEV process. Define a new process $\{y_t, t \geq 0\}$ by: $y_t = \frac{1}{\delta^2\beta^2} S_t^{-2\beta}$. By Ito's lemma, y_t follows a Feller (1951) diffusion:

$$dy_t = (ay_t + b)dt + 2\sqrt{y_t}\, dB_t, \tag{44}$$

where $a = -2\mu\beta$, $b = 2 + 1/\beta$. Further, take the square root of the process y_t: $z_t = \sqrt{y_t} = \frac{1}{\delta|\beta|} S_t^{-\beta}$. By Ito's lemma, the process $\{z_t, t \geq 0\}$ follows a *generalized Bessel diffusion*:

$$dz_t = \left(\frac{1+\beta}{2\beta z_t} - \mu\beta z_t \right) dt + \xi\, dB_t, \quad \xi = \mathrm{sign}(\beta). \tag{45}$$

This process is also known in the literature as the *radial Ornstein-Uhlenbeck process* (Shiga and Watanabe (1973), Eie (1983), Going-Jaeschke and Yor (1999)) and *Rayleigh process* (Giorno et al (1986)).

4.2 Double-Barrier Options

Consider a double-barrier call with two knock-out barriers L and U. To price this option, we need to compute the discounted risk-neutral expectation (4) with the underlying process (43). We will proceed according to the recipe of Section 3. The scale and speed densities of the CEV process are

$$\mathfrak{s}(S) = \exp\left(\frac{\mu}{\delta^2\beta}S^{-2\beta}\right), \quad \mathfrak{m}(S) = \frac{2}{\delta^2 S^{2+2\beta}}\exp\left(-\frac{\mu}{\delta^2\beta}S^{-2\beta}\right). \tag{46}$$

The speed density is used to define the inner product in the space of all square-integrable functions on $[L, U]$. To find explicit expressions for the eigenfunctions, we need to find $\xi_\lambda(S)$ and $\eta_\lambda(S)$ solving the initial value problems (33)-(35) with

$$\mathcal{A} = -\frac{1}{2}\delta^2 S^{2+2\beta}\frac{d^2}{dS^2} - \mu S\frac{d}{dS}. \tag{47}$$

Introduce a new variable

$$x := \frac{\mu}{\delta^2|\beta|}S^{-2\beta}. \tag{48}$$

We look for solutions to the ODE (33) with the CEV operator (47) in the form

$$u(S) = S^{\frac{1}{2}+\beta}e^{-\frac{x}{2}}w(x) \tag{49}$$

for some unknown function w. Substituting this functional form into Eq.(33), we arrive at the ODE for w:

$$\frac{d^2w}{dx^2} + \left(-\frac{1}{4} + \frac{k}{x} + \frac{\frac{1}{4} - m^2}{x^2}\right)w = 0, \quad x \in (l, u), \tag{50}$$

where the parameters k and m and the end-points of the interval l and u corresponding to the barriers L and U are:

$$m := \frac{1}{4|\beta|}, \quad k := m - \frac{1}{2} - \frac{\lambda}{2\mu|\beta|}, \quad l := \frac{\mu}{\delta^2|\beta|}L^{-2\beta}, \quad u := \frac{\mu}{\delta^2|\beta|}U^{-2\beta}. \tag{51}$$

This is the Whittaker's form of the confluent hypergeometric equation (Abramowitz and Stegun (1972), p. 505, and Slater (1960), p.9). Then the functions $\xi_\lambda(S)$ and $\eta_\lambda(S)$ can be written in the form

$$\xi_\lambda(S) = \frac{\delta^2}{2\mu}(SL)^{\frac{1}{2}+\beta}e^{\frac{l-x}{2}}g_k(x), \quad \eta_\lambda(S) = \frac{\delta^2}{2\mu}(SU)^{\frac{1}{2}+\beta}e^{\frac{u-x}{2}}h_k(x), \tag{52}$$

where $g_k(x)$ and $h_k(x)$ are unique solutions of the Whittaker equation (50) with the initial conditions

$$g_k(l) = 0, \quad g_k'(l) = 1, \quad h_k(u) = 0, \quad h_k'(u) = -1. \tag{53}$$

For any complex k, $m > 0$, and $a, b > 0$, introduce the following notation

$$\Delta_{k,m}(a,b) := W_{k,m}(a)W_{-k,m}(-b)e^{-ik\pi} - W_{k,m}(b)W_{-k,m}(-a)e^{-ik\pi}, \tag{54}$$

where $W_{k,m}(x)$ is the Whittaker function (Abramowitz and Stegun (1972), p. 505 and Slater (1960), p.10; Whittaker functions $M_{k,m}(x)$ and $W_{k,m}(x)$ are available in the *Mathematica* computer package). Then $g_k(x)$ and $h_k(x)$ can be taken in the form

$$g_k(x) = \Delta_{k,m}(l,x), \;\; h_k(x) = \Delta_{k,m}(x,u), \;\; x \in [l,u]. \tag{55}$$

Functions $W_{k,m}(x)$ and $W_{-k,m}(-x)$ provide two linearly independent solutions of the Whittaker equation (50) for any values of k and m (real or complex) with the Wronskian $e^{ik\pi}$ (Slater (1960), p.26, Eq.(2.4.31); $\epsilon = -1$ since we only consider Whittaker functions for real values of x). The Wronskian of $\eta_\lambda(S)$ and $\xi_\lambda(S)$ is given by Eq.(36) with the scale density of the CEV diffusion (46) and

$$C(\lambda) = \frac{\delta^2}{2\mu}(LU)^{\frac{1}{2}+\beta} e^{\frac{l+u}{2}} \Delta_{k,m}(l,u). \tag{56}$$

The eigenvalues ρ_n are found numerically as the negatives of zeros of $C(\lambda)$. Specifically, we need to find the roots $\{k_n, n = 1, 2, \dots\}$ of the equation

$$\Delta_{k,m}(l,u) = 0. \tag{57}$$

Then, from the second definition in Eq.(51), the eigenvalues ρ_n are

$$\rho_n = 2\mu|\beta|(k_n - m + \frac{1}{2}). \tag{58}$$

The roots of Eq.(57) are found numerically. Since, by Proposition 2, all ρ_n are positive, $k_n > m - \frac{1}{2}$ for all n. For each k_n, $g_{k_n}(x)$ and $h_{k_n}(x)$ are linearly dependent and

$$\xi_{-\rho_n}(S) = A_n \eta_{-\rho_n}(S), \;\; A_n = -e^{\frac{l-u}{2}} \left(\frac{L}{U}\right)^{\frac{1}{2}+\beta} \frac{W_{k_n,m}(l)}{W_{k_n,m}(u)}. \tag{59}$$

Then, by Eq.(41), the normalized eigenfunctions $\varphi_n(S)$ can be taken in the form

$$\varphi_n(S) = N_n\, S^{\frac{1}{2}+\beta} e^{-\frac{x}{2}} \Delta_{k_n,m}(l,x), \tag{60}$$

$$N_n = \sqrt{\frac{\delta^2|\beta| W_{k_n,m}(u)}{D_{n,m}(l,u) W_{k_n,m}(l)}}, \;\; D_{n,m}(l,u) = \left[\frac{\partial \Delta_{k,m}(l,u)}{\partial k}\right]\Bigg|_{k=k_n}. \tag{61}$$

For numerical calculations, two alternative representations of the function $\Delta_{k,m}(a,b)$ are useful (to prove use Eqs.(1.7.1), (1.7.7), (1.7.9), and (1.9.9) of Slater (1960)):

$$\Delta_{k,m}(a,b) = \frac{\pi}{\sin(2m\pi)} \left[\frac{M_{k,-m}(a)}{\Gamma(1-2m)} \frac{M_{k,m}(b)}{\Gamma(1+2m)} - \frac{M_{k,-m}(b)}{\Gamma(1-2m)} \frac{M_{k,m}(a)}{\Gamma(1+2m)}\right] \tag{62}$$

$$= \frac{\Gamma(\frac{1}{2}+m-k)}{\Gamma(1+2m)} \left[M_{k,m}(b) W_{k,m}(a) - M_{k,m}(a) W_{k,m}(b) \right], \tag{63}$$

where $\Gamma(x)$ is the Euler Gamma function (Abramowitz and Stegun (1972), p.255). For numerical computation of eigenvalues (Eq.(57)) and eigenfunctions (60) we used representation (63).

Proposition 3 *The double barrier call price under the CEV process is given by the eigenfunction expansion*

$$C(S,T,K,L,U) = e^{-rT} E_S[\mathbf{1}_{\{\mathcal{T}_{(L,U)}>T\}}(S_T-K)^+] = \sum_{n=1}^{\infty} e^{-(r+\rho_n)T} f_n \, \varphi_n(S), \tag{64}$$

with the eigenvalues (58), normalized eigenfunctions (60)-(61), and coefficients

$$f_n = N_n \frac{\Gamma(\frac{1}{2}+m-k_n)}{\Gamma(1+2m)} \left[W_{k_n,m}(l) I_n - M_{k_n,m}(l) J_n \right], \quad n = 1, 2, \dots, \tag{65}$$

$$I_n = \frac{1}{\delta\sqrt{\mu|\beta|}} \left[\frac{U^{\frac{1}{2}}}{2m+1} e^{\frac{u}{2}} M_{k_n+\frac{1}{2},m+\frac{1}{2}}(u) - \frac{2mKU^{-\frac{1}{2}}}{m-k_n-\frac{1}{2}} e^{\frac{u}{2}} M_{k_n+\frac{1}{2},m-\frac{1}{2}}(u) \right. \tag{66}$$
$$\left. - \frac{K^{\frac{1}{2}}}{2m+1} e^{\frac{\kappa}{2}} M_{k_n+\frac{1}{2},m+\frac{1}{2}}(\kappa) + \frac{2mK^{\frac{1}{2}}}{m-k_n-\frac{1}{2}} e^{\frac{\kappa}{2}} M_{k_n+\frac{1}{2},m-\frac{1}{2}}(\kappa) \right],$$

$$J_n = \frac{1}{\delta\sqrt{\mu|\beta|}} \left[\frac{U^{\frac{1}{2}}}{k_n+m+\frac{1}{2}} e^{\frac{u}{2}} W_{k_n+\frac{1}{2},m+\frac{1}{2}}(u) - \frac{KU^{-\frac{1}{2}}}{k_n-m+\frac{1}{2}} e^{\frac{u}{2}} W_{k_n+\frac{1}{2},m-\frac{1}{2}}(u) \right. \tag{67}$$
$$\left. - \frac{K^{\frac{1}{2}}}{k_n+m+\frac{1}{2}} e^{\frac{\kappa}{2}} W_{k_n+\frac{1}{2},m+\frac{1}{2}}(\kappa) + \frac{K^{\frac{1}{2}}}{k_n-m+\frac{1}{2}} e^{\frac{\kappa}{2}} W_{k_n+\frac{1}{2},m-\frac{1}{2}}(\kappa) \right],$$

where

$$\kappa := \frac{\mu}{\delta^2|\beta|} K^{-2\beta}. \tag{68}$$

The eigenfunction expansion (64) converges rapidly. Table 1 illustrates convergence of the eigenfunction expansion for double-barrier calls with one, three, and twelve months to expiration and $S = K = 100$, $L = 90$, $U = 120$, $r = 0.1$, $q = 0$. The CEV process parameters are selected in the following way. For each elasticity β ($\beta = 0, -0.5, -1, -2, -3, -4$), the scale parameter δ is selected so that the instantaneous volatility $\sigma(S) = \delta S^\beta$ is equal to 0.25 when $S = 100$ (see Boyle and Tian (1999) and Davydov and Linetsky (2000)). The values obtained by the numerical Laplace transform inversion in Davydov and Linetsky (2000) are provided for comparison. The agreement between the eigenfunction expansion and the numerical Laplace inversion is remarkable. For twelve months to expiration, only the first two or three terms in the eigenfunction expansion are needed to achieve the accuracy of five significant digits for double-barrier call prices. For three month options, three or four terms are needed. For one month options, five or six terms are needed.

	$\beta=0$	-0.5	-1	-2	-3	-4
N	Double-Barrier Call $T=1$ month					
1	4.8723	5.2942	5.7393	6.7041	7.7843	9.0208
2	3.0908	3.1116	3.1043	2.9892	2.7103	2.2176
3	2.9923	3.0536	3.1039	3.1770	3.2347	3.3242
4	3.0161	3.0834	3.1402	3.2237	3.2740	3.2999
5	3.0154	3.0820	3.1376	3.2169	3.2598	3.2764
6	3.0154	3.0820	3.1376	3.2171	3.2607	3.2794
Laplace	3.0154	3.0820	3.1376	3.2171	3.2606	3.2793
N	Double-Barrier Call $T=3$ months					
1	2.5586	2.8248	3.1157	3.7778	4.5590	5.4818
2	2.4135	2.6303	2.8579	3.3354	3.8193	4.2705
3	2.4131	2.6301	2.8579	3.3370	3.8268	4.2959
Laplace	2.4131	2.6300	2.8578	3.3370	3.8268	3.2959
N	Double-Barrier Call $T=12$ months					
1	0.1410	0.1672	0.1994	0.2859	0.4105	0.5827
2	0.1410	0.1672	0.1994	0.2859	0.4103	0.5822
Laplace	0.1410	0.1673	0.1994	0.2860	0.4104	0.5823

Table 1: Convergence of eigenfunction expansions for double-barrier call prices under the CEV processes with $\beta = 0, -0.5, -1, -2, -3, -4$ ($\beta = 0$ corresponds to the lognormal process) and $T = 1, 3$, and 12 months. For $T = 1$ month, for each price seven values are given: partial sums of the first N terms of the expansion Eq.(64) ($N = 1, ..., 6$) and the value obtained by the numerical Laplace inversion. For $T = 3$ months, for each price four values are given: partial sums of the first N terms of the expansion ($N = 1, 2, 3$) and the value obtained by the numerical Laplace inversion. For $T = 12$ months, for each price three values are given: partial sums of the first N terms of the expansion ($N = 1, 2$) and the value obtained by the numerical Laplace inversion. All numerical Laplace inversion values are taken from Davydov and Linetsky (2000). Parameters: $S = K = 100$, $L = 90$, $U = 120$, $r = 0.1$, $q = 0$, $\sigma(100) = 0.25$.

4.3 Up-and-Out Options

Consider an up-and-out call with some upper knock-out barrier U under the CEV process with $\beta < 0$ and $\mu > 0$. The payoff is $\mathbf{1}_{\{\mathcal{T}_U > T\}}(S_T - K)^+$, where $\mathcal{T}_U$ is the first hitting time of the upper barrier, $\mathcal{T}_U = \inf\{t : S_t = U\}$.

Proposition 4 *The up-and-out call price is given by the eigenfunction expansion*

$$C_{UO}(S,T,K,U) = e^{-rT} E_S[\mathbf{1}_{\{\mathcal{T}_U > T\}}(S_T - K)^+] = \sum_{n=1}^{\infty} e^{-(r+\rho_n)T} f_n \, \varphi_n(S) \tag{69}$$

with the eigenvalues $0 < \rho_1 < \rho_2 < \ldots < \rho_n < \ldots$, $\rho_n \to \infty$ *as* $n \to \infty$, *related by Eq.(58) to the roots* $\{k_n, n = 1, 2, \ldots\}$ *of the equation (to be solved numerically)*

$$M_{k,m}(u) = 0, \tag{70}$$

the corresponding normalized eigenfunctions

$$\varphi_n(S) = N_n \, S^{\frac{1}{2}+\beta} e^{-\frac{x}{2}} M_{k_n,m}(x), \tag{71}$$

$$N_n = \sqrt{\frac{\delta^2 |\beta| \Gamma(\frac{1}{2} + m - k_n) W_{k_n,m}(u)}{\Gamma(1+2m) \left[\frac{\partial M_{k,m}(u)}{\partial k}\right]\Big|_{k=k_n}}}, \tag{72}$$

and the coefficients

$$f_n = N_n \, I_n, \tag{73}$$

where I_n *are given by Eq.(66).*

Table 2 illustrates convergence of the eigenfunction expansion for the up-and-out call. The values obtained by the numerical Laplace transform inversion in Davydov and Linetsky (2000) are provided for comparison.

4.4 Down-and-Out Options

Consider a down-and-out call with some lower knock-out barrier L under the CEV process with $\beta < 0$ and $\mu > 0$. The payoff is $\mathbf{1}_{\{\mathcal{T}_L > T\}}(S_T - K)^+$, $\mathcal{T}_L = \inf\{t : S_t = L\}$. The domain of the problem is $[L, \infty)$. The associated Sturm-Liouville problem is *singular*. As discussed in the Proof of Proposition 4, it is *non-oscillatory*, and the spectrum is simple, purely discrete, and bounded below. The complication here is that the call payoff is *not* in $\mathcal{L}_2([L,\infty), \mathfrak{m})$ and, thus, the down-and-out call is *not* in the span of the $\mathcal{L}_2$-eigensecurities.[10] However, the down-and-out put payoff is in $\mathcal{L}_2([L,\infty), \mathfrak{m})$. We will price the down-and-out put first, and then find the price of the down-and-out call by appealing to a put-call parity result for down-and-out options.

[10] A similar situation when the payoff of a security to be priced is not in the Hilbert space of square-integrable payoffs was encountered in Lewis (1998).

	$\beta=-0.5$	-1	-2	-3	-4
N	Up-and-Out Call $T=1$ month				
1	0.0499	0.2883	1.4853	3.5512	6.2527
10	4.7555	3.1529	3.2334	3.2754	3.3011
20	3.0674	3.1445	3.2271	3.2757	3.3011
50	3.0870	3.1440	3.2271	3.2757	3.3011
Laplace	3.0870	3.1440	3.2271	3.2756	3.3010
N	Rebate $T=1$ month				
1	15.3320	12.5486	6.9059	2.0199	-2.2197
10	-1.3140	0.1604	0.0695	0.0307	0.0097
20	0.2389	0.1556	0.0755	0.0304	0.0097
50	0.2121	0.1561	0.0755	0.0304	0.0097
Laplace	0.2121	0.1561	0.0755	0.0304	0.0097
	Capped Call $T=1$ month				
	3.2928	3.3001	3.3026	3.3061	3.3108
N	Up-and-Out Call $T=12$ months				
1	0.0405	0.2006	0.7417	1.3290	1.8765
5	0.6591	0.8709	1.1246	1.4752	1.9061
10	0.7710	0.8709	1.1246	1.4752	1.9061
Laplace	0.7711	0.8709	1.1246	1.4752	1.9061
N	Rebate $T=12$ months				
1	15.5820	13.8024	11.7924	11.1851	10.9988
5	11.0576	10.8384	10.9107	10.9596	10.9617
10	10.7934	10.8384	10.9107	10.9596	10.9617
Laplace	10.7934	10.8384	10.9107	10.9596	10.9617
	Capped Call $T=12$ months				
	11.5645	11.7093	12.0353	12.4348	12.8678

Table 2: Convergence of eigenfunction expansions for up-and-out and capped call prices under the CEV processes with $\beta=-0.5,-1,-2,-3,-4$ and $T=1$ and 12 months. For $T=1$ month, for each up-and-out call five values are given: partial sums of the first N terms of the eigenfunction expansion Eq.(69) ($N=1,10,20,50$) and the value obtained by the numerical Laplace inversion. For each rebate five values are given: partial sums including the first N terms of the series in Eq.(83) ($N=1,10,20,50$) times the rebate amount $(U-K)$ and the value obtained by the numerical Laplace inversion. For $T=12$ months, for each up-and-out call four values are given: partial sums of the first N terms of the expansion ($N=1,5,10$) and the value obtained by the numerical Laplace inversion. For each rebate four values are given: partial sums including the first N terms of the series in Eq.(83) ($N=1,5,10$) times the rebate amount $(U-K)$ and the value obtained by the numerical Laplace inversion. Capped call prices are calculated according to (82) by adding the rebate to the up-and-out call. All numerical Laplace inversion values are taken from Davydov and Linetsky (2000). Parameters: $S=K=100$, $U=120$, $r=0.1$, $q=0$, $\sigma(100)=0.25$.

Proposition 5 *For any $L > 0$, the down-and-out put price is given by the eigenfunction expansion*

$$P_{DO}(S,T,K,L) = e^{-rT} E_S[\mathbf{1}_{\{\mathcal{T}_L > T\}}(K - S_T)^+] = \sum_{n=1}^{\infty} e^{-(r+\rho_n)T} f_n \, \varphi_n(S) \tag{74}$$

with the eigenvalues $0 < \rho_1 < \rho_2 < \ldots < \rho_n < \ldots$, $\rho_n \to \infty$ as $n \to \infty$, related by Eq.(58) to the roots $\{k_n, n = 1, 2, \ldots\}$ of the equation (to be solved numerically)

$$W_{k,m}(l) = 0, \tag{75}$$

the corresponding normalized eigenfunctions

$$\varphi_n(S) = N_n \, S^{\frac{1}{2}+\beta} e^{-\frac{x}{2}} W_{k_n,m}(x), \tag{76}$$

$$N_n = \sqrt{\frac{\delta^2 |\beta| \Gamma(\frac{1}{2} + m - k_n) M_{k_n,m}(l)}{\Gamma(1+2m) \left[\frac{\partial W_{k,m}(l)}{\partial k}\right]\Big|_{k=k_n}}}, \tag{77}$$

and the coefficients

$$f_n = \frac{N_n}{\delta\sqrt{\mu|\beta|}} \left[\frac{L^{\frac{1}{2}}}{k_n + m + \frac{1}{2}} e^{\frac{l}{2}} W_{k_n+\frac{1}{2}, m+\frac{1}{2}}(l) - \frac{K L^{-\frac{1}{2}}}{k_n - m + \frac{1}{2}} e^{\frac{l}{2}} W_{k_n+\frac{1}{2}, m-\frac{1}{2}}(l) \right. \tag{78}$$
$$\left. - \frac{K^{\frac{1}{2}}}{k_n + m + \frac{1}{2}} e^{\frac{\kappa}{2}} W_{k_n+\frac{1}{2}, m+\frac{1}{2}}(\kappa) + \frac{K^{\frac{1}{2}}}{k_n - m + \frac{1}{2}} e^{\frac{\kappa}{2}} W_{k_n+\frac{1}{2}, m-\frac{1}{2}}(\kappa) \right],$$

where κ is defined in Eq.(68).

Since the down-and-out call payoff is not in $\mathcal{L}_2([L,\infty),\mathfrak{m})$, it is not in the span of the eigenpayoffs (76). To price the down-and-out call, we first decompose its payoff as follows:

$$\mathbf{1}_{\{\mathcal{T}_L > T\}}(S_T - K)^+ = \mathbf{1}_{\{\mathcal{T}_L > T\}}(K - S_T)^+ + (S_T - K) - \mathbf{1}_{\{\mathcal{T}_L \le T\}}(S_T - K). \tag{79}$$

The first term on the right-hand side is the payoff of a down-and-out put, the second term is the payoff from a forward contract with the delivery price K, and the last term $\mathbf{1}_{\{\mathcal{T}_L \le T\}}(S_T - K)$ can be interpreted as a *down-and-in forward contract* that is activated if and only if the underlying asset price hits the lower barrier L prior to and including maturity T and pays the amount equal to $(S_T - K)$ at T if activated. Taking the present values of both sides of the equality (79), we have for the prices at $t = 0$:

$$C_{DO} = P_{DO} + (e^{-qT} S - e^{-rT} K) - f_{DI}. \tag{80}$$

We have already priced the down-and-out put P_{DO}. To price the down-and-out call C_{DO} we need to price the down-and-in forward contract f_{DI}.

Proposition 6 *The price of a down-and-in forward contract under the CEV process with $\beta < 0$ and $\mu > 0$ is $(S \geq L > 0)$:*

$$f_{DI} = e^{-rT} E_S[\mathbf{1}_{\{\mathcal{T}_L \leq T\}}(S_T - K)] = e^{-qT} S \left(\frac{G(-2m, x)}{G(-2m, l)} \right) - e^{-rT} K \left(\frac{G(2m, x)}{G(2m, l)} \right) \quad (81)$$

$$+ \sum_{n=1}^{\infty} e^{-(r+\rho_n)T} \left[\frac{L}{\mu + \rho_n} - \frac{K}{\rho_n} \right] \frac{2\mu|\beta| S^{\frac{1}{2}+\beta} e^{-\frac{x}{2}} W_{k_n,m}(x)}{L^{\frac{1}{2}+\beta} e^{-\frac{l}{2}} \left[\frac{\partial W_{k,m}(l)}{\partial k} \right] \Big|_{k=k_n}},$$

where ρ_n are the eigenvalues of the Sturm-Liouville problem on $[L, \infty)$, k_n are the roots of the equation (75), x is defined in Eq.(48), l is defined in Eq.(51), and $G(\nu, a)$ is the complementary Gamma distribution function, $G(\nu, a) = \frac{1}{\Gamma(\nu)} \int_a^{\infty} e^{-t} t^{\nu-1} dt$.

Now we can compute the down-and-out call price using the put-call parity relationship (80). Table 3 illustrates convergence of the eigenfunction expansion for the down-and-out call. The values obtained by the numerical Laplace transform inversion in Davydov and Linetsky (2000) are provided for comparison.

4.5 Capped Options

In addition to their popularity over-the-counter, several types of barrier options are traded on securities exchanges. Capped call (and put) options on the S&P 100 and S&P 500 indices were introduced by the Chicago Board of Options Exchange (CBOE) in November, 1991. A *capped call* is an up-and-out call with the cash rebate equal to the difference between the upper barrier (*cap*) and the strike price (see Broadie and Detemple (1995)). It combines a European exercise feature and an automatic exercise feature. The automatic exercise is triggered when the index value first exceeds the cap (the rebate equal to the intrinsic value of the call is paid at the time the index first exceeds the cap). The price of a capped call can be represented as the sum of the up-and-out call price and the price of rebate (U is the cap price):

$$CappedCall(S, T, K, U) = C_{UO}(S, T, K, U) + (U - K) E_S[e^{-r\mathcal{T}_U} \mathbf{1}_{\{\mathcal{T}_U \leq T\}}]. \quad (82)$$

We have already priced the up-and-out call in Section 4.3. To price capped calls, we need to evaluate the price of rebate. In Davydov and Linetsky (2000) the price of rebate was expressed as the inverse Laplace transform of a known function (Eqs.(4), (16), and (37)). Here we invert the Laplace transform by applying the methods developed in this paper. Table 2 illustrates convergence of the series (83). The values obtained by the numerical Laplace inversion in Davydov and Linetsky (2000) are provided for comparison.

Proposition 7 *Under the CEV process with $\beta < 0$ and $\mu > 0$, the price of the rebate is:*

$$E_S[e^{-r\mathcal{T}_U} \mathbf{1}_{\{\mathcal{T}_U \leq T\}}] = \frac{S^{\frac{1}{2}+\beta} e^{-\frac{x}{2}}}{U^{\frac{1}{2}+\beta} e^{-\frac{u}{2}}} \left\{ \frac{M_{\varkappa,m}(x)}{M_{\varkappa,m}(u)} + \sum_{n=1}^{\infty} \frac{2\mu|\beta| e^{-(r+\rho_n)T} M_{k_n,m}(x)}{(r + \rho_n) \left[\frac{\partial M_{k,m}(u)}{\partial k} \right] \Big|_{k=k_n}} \right\}, \quad (83)$$

	$\beta = -0.5$	-1	-2	-3	-4
N	Down-and-Out Put $T = 3$ months				
10	0.0754	0.1388	0.2028	0.2139	0.2013
50	0.3179	0.3769	0.3330	0.2788	0.2336
150	0.4352	0.4006	0.3338	0.2788	0.2336
250	0.4392	0.4006	0.3338	0.2788	0.2336
Laplace	0.4391	0.4005	0.3337	0.2788	0.2336
N	Down-and-In Forward $T = 3$ months				
1	9.7751	7.2079	4.2073	2.9261	2.3094
5	9.7938	7.4046	4.7494	3.6183	3.0442
10	9.7938	7.4047	4.7496	3.6185	3.0446
	Down-and-Out Call $T = 3$ months				
$N = 250$	5.9609	5.9336	5.8791	5.8246	5.7704
Laplace	5.9608	5.9336	5.8790	5.8246	5.7704
N	Down-and-Out Put $T = 12$ months				
10	0.0381	0.0474	0.0411	0.0316	0.0241
50	0.0642	0.0558	0.0419	0.0317	0.0241
75	0.0646	0.0558	0.0419	0.0317	0.0241
Laplace	0.0646	0.0558	0.0419	0.0317	0.0241
N	Down-and-In Forward $T = 12$ months				
1	4.7053	2.8181	1.0643	0.5838	0.4707
5	4.7185	2.9328	1.2893	0.7919	0.6319
10	4.7185	2.9328	1.2893	0.7919	0.6319
	Down-and-Out Call $T = 12$ months				
$N = 75$	11.2354	11.1540	11.0086	10.8821	10.7713
Laplace	11.2354	11.1540	11.0086	10.8821	10.7713

Table 3: Convergence of eigenfunction expansions for down-and-out put and call prices under the CEV processes with $\beta = -0.5, -1, -2, -3, -4$ and $T = 3$ and 12 months. For $T = 3$ months, for each down-and-out put five values are given: partial sums of the first N terms of the eigenfunction expansion Eq.(74) ($N = 10, 50, 150, 250$) and the value obtained by the numerical Laplace inversion. For each down-and-in forward three values are given: partial sums including the first N terms in the series in Eq.(81) ($N = 1, 5, 10$). For $T = 12$ months, for each down-and-out put four values are given: partial sums of the first N terms of the series Eq.(74) ($N = 10, 50, 75$) and the value obtained by the numerical Laplace inversion. For each down-and-in forward three values are given: partial sums including the first N terms of the series in Eq.(81) ($N = 1, 5, 10$). Down-and-out call prices are calculated according to the put-call parity relationship for the down-and-out options Eq.(80). N indicates the number of terms taken in the eigenfunction expansion for the down-and-out put. All numerical Laplace inversion values are taken from Davydov and Linetsky (2000). Parameters: $S = K = 100$, $L = 90$, $r = 0.1$, $q = 0$, $\sigma(100) = 0.25$.

where $0 < S < U$, k_n *are the roots of the equation (70),* ρ_n *are the eigenvalues of the Sturm-Liouville problem on* $[0, U]$ *related to* k_n *by Eq.(58),* x *is defined in Eq.(48),* u *is defined in Eq.(51), and* $\varkappa := m - \frac{1}{2} - \frac{r}{2\mu|\beta|}$.

4.6 Vanilla Options

Now consider the problem of pricing vanilla options (without barriers) under the CEV process with $\beta < 0$ and $\mu > 0$. The domain of the problem is $[0, \infty)$. The associated Sturm-Liouville problem is singular and, as discussed in the Proof of Proposition 4, non-oscillatory. The spectrum is simple, purely discrete, and bounded below. As for the down-and-out call, the vanilla call payoff is not in $\mathcal{L}_2([0, \infty), \mathfrak{m})$ and, thus, the vanilla call is not in the span of the $\mathcal{L}_2$-eigensecurities. However, the put payoff is in $\mathcal{L}_2([0, \infty), \mathfrak{m})$. We will price the put first, and then find the price of the call by appealing to the put-call parity.

Proposition 8 *(i) The spectral resolution of the continuous transition probability density for the CEV process on* $[0, \infty)$ *with* $\beta < 0$ *and* $\mu > 0$ *is:*

$$p(T; S, S_T) = \mathfrak{m}(S_T) \sum_{n=1}^{\infty} e^{-\rho_n T} \varphi_n(S) \varphi_n(S_T), \tag{84}$$

where the eigenvalues and the corresponding normalized eigenfunctions are $(n = 1, 2, \dots)$

$$\rho_n = 2\mu|\beta|n, \quad \varphi_n(S) = N_n S e^{-x} L_{n-1}^{2m}(x), \quad N_n = \sqrt{\frac{(n-1)!\mu}{\Gamma(2m+n)}} \left(\frac{\mu}{\delta^2|\beta|}\right)^m, \tag{85}$$

where $L_n^\nu(x)$ *are the generalized Laguerre polynomials (Abramowitz and Stegun (1972)).*

(ii) The probability of absorption at zero is given by:

$$\Pr(S_T = 0 | S_0 = S) = G\left(2m, \frac{x}{1 - e^{2\mu\beta T}}\right), \tag{86}$$

where $G(\nu, a)$ *is the complementary Gamma distribution function.*

(iii) The price of the plain vanilla put is given by the eigenfunction expansion

$$P(S, T, K) = e^{-rT} K G\left(2m, \frac{x}{1 - e^{2\mu\beta T}}\right) + \sum_{n=1}^{\infty} e^{-(r+\rho_n)T} f_n \varphi_n(S) \tag{87}$$

with the coefficients

$$f_n = \frac{N_n K}{\mu} \left[\frac{\Gamma(2m+n)}{\Gamma(2m) n!} - L_n^{2m-1}(\kappa) - \frac{\kappa}{2m+n} L_{n-1}^{2m+1}(\kappa)\right], \tag{88}$$

where κ *is defined in Eq.(68). The price of the plain vanilla call is found from the put-call parity relationship* $C(S, T, K) = P(S, T, K) + e^{-qT} S - e^{-rT} K$.

Thus we have obtained the eigenfunction expansion for vanilla options under the CEV process. What is the relationship of our result with the classic CEV option pricing formula of Cox (1975)? Cox's formula expresses the CEV option prices in terms of the complementary non-central chi-square distribution function, while our formula expresses option prices as infinite sums of Laguerre polynomials. The equivalence is established by appealing to the *Hille-Hardy formula* (Erdelyi (1955), Vol.II, p.189) (for all $|t| < 1$, $\nu > -1$, $a, b > 0$)

$$(ab)^{\frac{\nu}{2}} \sum_{n=0}^{\infty} \frac{t^{n+\frac{\nu}{2}} n!}{\Gamma(n+\nu+1)} L_n^{\nu}(a) L_n^{\nu}(b) = \frac{1}{1-t} \exp\left\{-\frac{(a+b)t}{1-t}\right\} I_{\nu}\left(\frac{2\sqrt{tab}}{1-t}\right), \tag{89}$$

where $I_{\nu}(a)$ is the modified Bessel function of the first kind. Applying this summation formula to the spectral resolution (84) and identifying $t = e^{2\mu\beta T}$ yields the standard form of the continuous CEV density used by Cox (1975) ($x = \frac{\mu}{\delta^2|\beta|} S^{-2\beta}$, $x_T = \frac{\mu}{\delta^2|\beta|} S_T^{-2\beta}$):

$$p(T; S, S_T) = \frac{2\mu S_T^{-2\beta-\frac{3}{2}} S^{\frac{1}{2}}}{\delta^2(e^{-2\mu\beta T} - 1)} \exp\left(\frac{x_T + xe^{-2\mu\beta T}}{1 - e^{-2\mu\beta T}} + \frac{\mu T}{2}\right) I_{\frac{1}{2|\beta|}}\left(\frac{\sqrt{x x_T}}{\sinh(\mu|\beta|T)}\right). \tag{90}$$

Integrating this density with the option payoff leads to Cox's formula expressed in terms of the complementary chi-square distribution function (see Schroder (1989) and Davydov and Linetsky (2000)).

Table 4 illustrates convergence of the series for the vanilla call. The values obtained by computing Cox's formula (1975) are provided for comparison (we use the algorithm provided by Schroder (1989)). The convergence for vanilla options is slower than for double-barrier options since the eigenvalues in (85) grow linearly with n, in contrast to the n^2 growth for the regular Sturm-Liouville problem with two barriers.

5 Interest Rate Knock-Out Options in the CIR Term Structure Model

5.1 The CIR Process

In this Section we consider interest rate options with barriers. A zero-coupon *knock-out bond* pays one dollar at maturity $T > 0$ if some reference interest rate (e.g., three-month LIBOR) does not leave a pre-specified range (corridor) prior to maturity, and zero otherwise.

Suppose that under the risk-neutral measure Q the instantaneous risk-free interest rate follows the Cox-Ingersoll-Ross (CIR) diffusion process on $(0, \infty)$

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}\, dB_t, \quad t > 0, \quad r_0 = r > 0, \tag{91}$$

where $\{B_t, t \geq 0\}$ is a standard Brownian motion, $\theta > 0$ is the long-run level, $\kappa > 0$ is the rate of mean reversion to the long run level, $\sigma > 0$ is the volatility parameter, and

	$\beta = -0.5$	-1	-2	-3	-4
N	Vanilla Call $T = 1$ month				
10	10.1991	3.9635	2.6486	4.1398	4.5958
100	4.6459	3.1336	3.2952	3.3072	3.3105
200	2.9938	3.3218	3.3027	3.3063	3.3109
500	3.3261	3.3011	3.3031	3.3063	3.3109
1000	3.3005	3.3012	3.3031	3.3063	3.3109
Cox	3.3005	3.3012	3.3031	3.3063	3.3109
N	Vanilla Call $T = 12$ months				
2	17.9452	20.2703	16.1075	15.2465	15.3931
10	17.0308	14.7942	15.0904	15.2647	15.4770
50	14.9762	15.0022	15.0892	15.2619	15.4767
100	14.9824	15.0022	15.0892	15.2619	15.4767
Cox	14.9824	15.0022	15.0892	15.2619	15.4767

Table 4: Convergence of eigenfunction expansions for vanilla call prices under the CEV processes with $\beta = -0.5, -1, -2, -3, -4$ and $T = 1$ and 12 months. For $T = 1$ month, for each price six values are given: partial sums of the first N terms of the expansion ($N = 10, 100, 200, 500, 1000$) and the value obtained by Cox's formula. For $T = 12$ months, for each price five values are given: partial sums of the first N terms of the expansion ($N = 2, 10, 50, 100$) and the value obtained by Cox's formula. Parameters: $S = K = 100$, $r = 0.1$, $q = 0$, $\sigma(100) = 0.25$.

the initial interest rate is $r > 0$. To insure that the origin is inaccessible (the short rate stays strictly positive), the parameters are assumed to satisfy Feller's condition $2\kappa\theta \geq \sigma^2$. For this choice of parameters, the origin is an entrance boundary and infinity is a natural boundary. CIR (1985) derive a closed-form expression for the time $t = 0$ price of a zero-coupon bond that pays one dollar at maturity $T > 0$:

$$P(r, T) = E_r\left[e^{-\int_0^T r(t)\,dt}\right] = A(T)e^{-B(T)r}, \tag{92}$$

where

$$A(T) := \left(\frac{2\gamma e^{(\kappa+\gamma)T/2}}{(\gamma+\kappa)(e^{\gamma T}-1)+2\gamma}\right)^b, \quad B(T) := \frac{2(e^{\gamma T}-1)}{(\gamma+\kappa)(e^{\gamma T}-1)+2\gamma}, \tag{93}$$

$$\gamma := \sqrt{\kappa^2 + 2\sigma^2}, \quad b := \frac{2\kappa\theta}{\sigma^2}. \tag{94}$$

CIR also derive closed-form expressions for European call and put options on zero-coupon bonds.

5.2 Eigenfunction Expansions for Knock-Out Bonds

In this section we focus on pricing knock-out contracts where the reference interest rate is the LIBOR rate $L(t, t+\delta)$ (e.g., three-month LIBOR). These contracts knock out when the LIBOR leaves some pre-specified corridor $(\underline{L}, \overline{L})$. In the CIR model there is an analytical one-to-one relationship between the LIBOR rate[11] and the short rate:

$$L(t, t+\delta) = \frac{1}{\delta}[A^{-1}(\delta) e^{B(\delta) r_t} - 1]. \tag{95}$$

Then the event $\{$*LIBOR leaves the corridor* $(\underline{L}, \overline{L})\}$ is equivalent to the event $\{$*short rate leaves the corridor* $(L, U)\}$, where

$$L = B^{-1}(\delta) \ln(A(\delta)(1 + \delta \underline{L})), \;\; U = B^{-1}(\delta) \ln(A(\delta)(1 + \delta \overline{L})). \tag{96}$$

To price a zero-coupon knock-out bond that is knocked out when the LIBOR rate exits the corridor $(\underline{L}, \overline{L})$ (short rate exits the corridor (L, U)), we need to evaluate the expectation

$$P(r, T, L, U) = E_r \left[e^{-\int_0^T r(t) dt} \mathbf{1}_{\{\mathcal{T}_{(L,U)} > T\}} \right], \tag{97}$$

where $\mathcal{T}_{(L,U)} = \inf\{t : r_t \notin (L, U)\}$.

The speed density of the CIR diffusion is (b is defined in Eq.(94))

$$\mathfrak{m}(r) = \frac{2}{\sigma^2} r^{b-1} \exp\left(-\frac{2\kappa r}{\sigma^2} \right) \tag{98}$$

and is used to define the inner product in the space of all square-integrable functions on $[L, U]$. To find explicit expressions for the eigenfunctions, we need to find the functions $\xi_\lambda(r)$ and $\eta_\lambda(r)$ solving the initial value problems (33)-(35) with the CIR operator

$$\mathcal{A} = -\frac{1}{2}\sigma^2 r \frac{d^2}{dr^2} - \kappa(\theta - r)\frac{d}{dr} + r. \tag{99}$$

Introduce a new variable

$$x := \frac{2\gamma r}{\sigma^2}. \tag{100}$$

We look for solutions to the ODE (33) with the CIR operator (99) in the form

$$u(r) = r^{-\frac{b}{2}} \exp\left(\frac{\kappa r}{\sigma^2} \right) w(x) \tag{101}$$

for some unknown function $w(x)$. Substituting this functional form into the ODE, we arrive at the Whittaker equation (50) with the parameters k and m and end-points l and u of the interval corresponding to short-rate barriers L and U given by:

$$m := \frac{b-1}{2}, \;\; k := \frac{1}{\gamma}\left(\frac{\kappa b}{2} - \lambda \right), \;\; l := \frac{2\gamma L}{\sigma^2}, \;\; u := \frac{2\gamma U}{\sigma^2}. \tag{102}$$

[11]Recall that the LIBOR $L(t, t+\delta)$ is a simple interest rate for the period $[t, t+\delta]$, and $L(t, t+\delta) = \frac{1}{\delta}\left(\frac{1}{P(t, t+\delta)} - 1 \right)$, where $P(t, t+\delta)$ is the time-t price of a zero-coupon bond with unit face and maturity $t + \delta$.

Then, from the second definition in Eq.(102), the eigenvalues $\{\rho_n, n = 1, 2, \dots\}$ for this problem are

$$\rho_n = \gamma k_n - \frac{\kappa b}{2}, \tag{103}$$

where k_n are the roots of Eq.(57). The corresponding normalized eigenfunctions are:

$$\varphi_n(r) = N_n r^{-\frac{b}{2}} \exp\left(\frac{\kappa r}{\sigma^2}\right) \Delta_{k_n,m}(l, x), \tag{104}$$

$$N_n = \sqrt{\frac{\sigma^2 W_{k_n,m}(u)}{2D_{n,m}(l,u) W_{k_n,m}(l)}}, \tag{105}$$

where $D_{n,m}(l,u)$ is defined in Eq.(61).

Finally, the knock-out bond price is given by the eigenfunction expansion

$$P(r, T, L, U) = \sum_{n=1}^{\infty} e^{-\rho_n T} f_n \, \varphi_n(r) \tag{106}$$

with the coefficients (in the case of knock-out bonds the payoff function is $f(r_T) = 1$)

$$f_n = (1, \varphi_n) = \int_L^U \varphi_n(y) \mathfrak{m}(y) dy, \;\; n = 1, 2, \dots . \tag{107}$$

In contrast with the case of double-barrier options under the CEV process, the integrals in Eq.(107) cannot be calculated analytically and must be computed numerically.

Table 5 illustrates convergence of the series (106) for knock-out bonds with $T = 0.5$, 1, 3, 5, and 10 years to maturity. The CIR process parameters are $\theta = 0.07$, $\kappa = 0.2$, $\sigma = 0.1$. The initial short rate is $r = 0.06$, and the lower and upper barriers are $\underline{L} = 0.02$ and $\overline{L} = 0.11$. The series converges rapidly. For five and ten years only the first term is needed to achieve the accuracy of five significant digits. For shorter maturities more terms are needed. For comparison, Table 5 also gives vanilla zero-coupon CIR bonds prices and yields. The spread compensates for the risk of knock-out.

More generally, any interest rate derivative with some interest rate dependent payoff at maturity and knock-out barriers can be priced by the eigenfunction expansion method. For example, a *knock-out cap* is a cap that knocks out at the first time the LIBOR leaves the corridor $(\underline{L}, \overline{L})$ (all remaining caplets are extinguished as soon as either $\underline{L}$ or $\overline{L}$ is hit). An individual caplet pays an amount $\delta(L(T, T+\delta) - K)^+$ at time $T+\delta$, where $L(T, T+\delta)$ is the LIBOR for the period $[T, T+\delta]$ observed at time T (see Hull (2000)). A *knock-out caplet* payoff is $\delta(L(T, T+\delta) - K)^+ \mathbf{1}_{\{\mathcal{T}_{(\underline{L},\overline{L})} > T\}}$, $\mathcal{T}_{(\underline{L},\overline{L})} = \inf\{t : L(t, t+\delta) \notin (\underline{L}, \overline{L})\}$. This time-$(T+\delta)$ cash flow is equivalent to a time-T cash flow:

$$\frac{\delta(L(T, T+\delta) - K)^+}{1 + \delta L(T, T+\delta)} \mathbf{1}_{\{\mathcal{T}_{(\underline{L},\overline{L})} > T\}} = (1 - (1 + \delta K) P(T, T+\delta))^+ \mathbf{1}_{\{\mathcal{T}_{(\underline{L},\overline{L})} > T\}}$$

	$T = 0.5$	1	3	5	10
N	Knock-out Bond Price				
1	1.0012	0.8671	0.4878	0.2745	0.0652
2	1.0145	0.8741	0.4884	0.2745	0.0652
3	0.9573	0.8608	0.4883	0.2745	0.0652
4	0.9561	0.8607	0.4883	0.2745	0.0652
5	0.9578	0.8608	0.4883	0.2745	0.0652
6	0.9579	0.8608	0.4883	0.2745	0.0652
	Knock-out Bond Yield				
	0.0861	0.1499	0.2389	0.2586	0.2731
	Vanilla Bond Price				
	0.9702	0.9410	0.8306	0.7320	0.5334
	Vanilla Bond Yield				
	0.0605	0.0608	0.0619	0.0624	0.0628

Table 5: Convergence of eigenfunction expansions for zero-coupon double knock-out bonds with maturities $T = 0.5, 1, 3, 5, 10$ years in the CIR term structure model. Each bond pays one dollar at maturity T if the three-month LIBOR rate never leaves the corridor (0.02,0.11) during the lifetime of the bond, and zero otherwise. For each bond, six values are given: partial sums of the first N terms in the eigenfunction expansion ($N = 1, ..., 6$). Parameters: $\kappa = 0.2$, $\sigma = 0.1$, $\theta = 0.07$, $r_0 = 0.06$. Corresponding vanilla bond prices and yields are given for comparison.

$$= (1 - (1 + \delta K)A(\delta)e^{-B(\delta)r_T})^+ \mathbf{1}_{\{\mathcal{T}_{(L,U)} > T\}},$$

$\mathcal{T}_{(L,U)} = \inf\{t : r_T \notin (L, U)\}$. The present value of this cash flow at $t = 0$ is given by the eigenfunction expansion of the form Eq.(106) with the coefficients

$$f_n = \int_{r^*}^{U} [1 - (1 + \delta K)A(\delta)e^{-B(\delta)y}]\varphi_n(y)\mathfrak{m}(y)dy, \quad n = 1, 2, \dots , \tag{108}$$

$$r^* = B^{-1}(\delta)\ln(A(\delta)(1 + \delta K)). \tag{109}$$

In this Section we focused on the double-barrier (corridor) interest rate options. Single-barrier up-and-out and down-and-out interest rate options can be priced similarly to the single-barrier CEV options of Sections 4.3 and 4.4 by solving singular Sturm-Liouville problems for the Whittaker equation on the intervals $(0, u]$ and $[l, \infty)$.

5.3 Eigenfunction Expansions for Vanilla Zero-Coupon Bonds

Consider again a problem of pricing vanilla zero-coupon bonds (no barriers). The solution is given by the CIR formula (92)-(94). One may wonder how does the CIR formula emerge in the eigenfunction expansion framework.

Proposition 9 *(i) The spectral resolution of the state-price density in the CIR model is:*

$$p(T; r, r_T) = \mathfrak{m}(r_T) \sum_{n=1}^{\infty} e^{-\rho_n T} \varphi_n(r) \varphi_n(r_T), \tag{110}$$

where the eigenvalues and the corresponding normalized eigenfunctions are $(n = 1, 2, ...)$

$$\rho_n = \gamma(n-1) + \frac{b}{2}(\gamma - \kappa), \tag{111}$$

$$\varphi_n(r) = N_n e^{\frac{(\kappa-\gamma)r}{\sigma^2}} L_{n-1}^{b-1}(x), \quad N_n = \sqrt{\frac{\sigma^2\,(n-1)!}{2\Gamma(b+n-1)}} \left(\frac{2\gamma}{\sigma^2}\right)^{\frac{b}{2}}, \tag{112}$$

where $L_n^{\nu}(x)$ *are the generalized Laguerre polynomials and* $x = \frac{2\gamma r}{\sigma^2}$.

(ii) The zero-coupon bond price is given by the eigenfunction expansion:

$$P(r, T) = \sum_{n=1}^{\infty} e^{-\rho_n T} f_n \, \varphi_n(r), \tag{113}$$

$$f_n = \frac{2N_n \Gamma(b+n-1)}{\sigma^2\,(n-1)!} \left(\frac{\sigma^2}{\gamma+\kappa}\right)^b \left(\frac{\kappa-\gamma}{\kappa+\gamma}\right)^{n-1}. \tag{114}$$

The claims with payoffs $\varphi_n(r_T)$ form a complete set of eigensecurities in the space of all T-maturity $\mathcal{L}_2((0,\infty);\mathfrak{m})$-claims in the CIR economy. The expression (113) unbundles the zero-coupon bond into a portfolio of eigensecurities. Finally, the CIR bond pricing formula (92)-(94) is recovered by performing the summation in Eq.(113) using the identity (Gradshteyn and Ryzhik (1994), p.1063)(for all $|z| < 1$, $\alpha > -1$)

$$\sum_{n=0}^{\infty} z^n L_n^{\alpha}(x) = (1-z)^{-\alpha-1} \exp\left(\frac{xz}{z-1}\right), \tag{115}$$

and identifying $z = e^{-\gamma T}\left(\frac{\kappa-\gamma}{\kappa+\gamma}\right)$ in Eq.(113).

6 Conclusion

This paper develops an eigenfunction expansion approach to pricing options on scalar diffusion processes. All European-style contingent claims with payoffs square-integrable with the speed measure are unbundled into portfolios of primitive securities termed *eigensecurities*. Eigensecurities are eigenvectors of the pricing operator and are fundamental building blocks in our approach. All other European-style contingent claims are represented as portfolios of eigensecurities. In particular, Arrow-Debreu securities themselves are unbundled into portfolios of eigensecurities. This produces an eigenfunction expansion of the state-price density termed *spectral resolution of the state-price density.*

In this paper we show that the eigenfunction expansion method is a powerful computational tool for derivatives pricing. While the state-price density solves the initial- and boundary-value problem for the pricing PDE, the eigensecurities are solutions to the *static* pricing equation without the time derivative term. This static pricing equation can be interpreted as a second-order Sturm-Liouville ODE. The rich theory of the Sturm-Liouville equation can then be applied to derivatives pricing.

To illustrate the computational power of the method, this paper develops two specific applications: pricing vanilla, single- and double-barrier options under the CEV process and interest rate knock-out options in the CIR term structure model. For the CEV process, our main result is the analytical inversion of the Laplace transforms in maturity for single- and double-barrier options in Davydov and Linetsky (2000). For the CIR process, we derive analytical expressions for the prices of knock-out bonds. In both applications, the eigenfunction expansions converge rapidly, with the first several terms sufficient to insure excellent accuracy. Further applications of eigenfunction expansions to problems in financial engineering will be explored in future research.

A Proofs

Proof of Proposition 1. Introduce a new variable $x := \frac{1}{\sigma}\ln\left(\frac{S}{L}\right)$. It is convenient to work with the Brownian motion B_t^x starting at x at $t=0$, $B_t^x = B_t + x$ $(0 < x < u)$. In terms of B_t^x, the process (1) can be represented in the form $S_t = Le^{\sigma(B_t^x + \nu t)}$, $t \geq 0$. Due to the Cameron-Martin-Girsanov theorem, we have

$$E_S[\mathbf{1}_{\{\mathcal{T}_{(L,U)}>T\}}\,\varphi_n(S_T)] = E_x[e^{\nu(B_T^x - x) - \nu^2 T/2}\,\mathbf{1}_{\{\Sigma_{(0,u)}>T\}}\varphi_n(Le^{\sigma B_T^x})]$$

$$= e^{-\frac{\nu^2 T}{2}}\sqrt{\frac{\sigma}{u}}\,S^{-\frac{\nu}{\sigma}}\,E_x\left[\mathbf{1}_{\{\Sigma_{(0,u)}>T\}}\sin\left(\frac{\pi n}{u}B_T^x\right)\right], \tag{116}$$

where $\Sigma_{(0,u)} = \inf\{t : B_t^x \notin (0,u)\}$. Define

$$V(x,T) := E_x\left[\mathbf{1}_{\{\Sigma_{(0,u)}>T\}}\sin\left(\frac{\pi n}{u}B_T^x\right)\right],\ \ x \in [0,u],\ \ T \in [0,\infty). \tag{117}$$

The function $V(x,T)$ is a unique continuous solution of the standard heat equation with absorbing boundary conditions at 0 and u and the initial condition at $T=0$ (see Karatzas and Shreve (1991), pp. 266-7):

$$\frac{1}{2}\frac{\partial^2 V}{\partial x^2} = \frac{\partial V}{\partial T},\ \ x \in (0,u),\ \ T \in (0,\infty), \tag{118}$$

$$V(0,T) = 0,\ \ V(u,T) = 0,\ \ T \in [0,\infty),$$

$$V(x,0) = \sin\left(\frac{\pi n x}{u}\right),\ \ x \in (0,u).$$

The function $V(x,T) = e^{-\left(\frac{n^2\pi^2}{2u^2}\right)T} \sin\left(\frac{\pi n}{u}x\right)$ uniquely solves this problem. Thus, by definition (117), we have

$$E_x \left[\mathbf{1}_{\{\Sigma_{(0,u)} > T\}} \sin\left(\frac{\pi n}{u} B_T^x\right)\right] = e^{-\left(\frac{n^2\pi^2}{2u^2}\right)T} \sin\left(\frac{\pi n x}{u}\right). \tag{119}$$

Substituting this result into Eq.(116) and recalling the definition (8), we prove Eqs.(10), (11). Thus, the functions φ_n are eigenvectors of the pricing operator. This proves part (i) of the Proposition. Part (ii) follows from the standard facts of Fourier analysis. Recall that the system $\{\sqrt{\frac{2}{\pi}}\sin(ny), n = 1, 2, ...\}$ is complete and orthonormal in $\mathcal{L}_2[0,\pi]$ with the inner product $\int_0^\pi f(y)g(y)dy$. Changing the variable $y = \frac{\pi}{\sigma u}\ln\left(\frac{X}{L}\right)$ and pre-multiplying the functions by $\sqrt{\frac{\pi\sigma}{2u}}X^{-\frac{\nu}{\sigma}}$, the system $\{\sqrt{\frac{\sigma}{u}}X^{-\frac{\nu}{\sigma}} \sin\left[\frac{\pi n}{\sigma u}\ln\left(\frac{X}{L}\right)\right],\ n = 1, 2, ...\}$ is complete and orthonormal in $\mathcal{L}_2([L,U],\mathfrak{m})$ with the inner product (7). Then any $\mathcal{L}_2$ payoff is in the span of φ_n, and the Fourier coefficients f_n are determined by calculating the inner product (14). The convergence is in the norm of the Hilbert space. Finally, the pricing formula (15) follows from the payoff decomposition (13), the linearity of the pricing operator, and the eigenvector property of the eigenpayoffs φ_n (10).□

Proof of Corollary 1. For the double barrier call payoff we have:

$$f_n = \int_K^U (X-K)\varphi_n(X)\mathfrak{m}(X)dX = \frac{2}{\sigma\sqrt{\sigma u}}\int_K^U (X-K)X^{\frac{\nu}{\sigma}} \sin\left[\frac{\pi n}{\sigma u}\ln\left(\frac{X}{L}\right)\right]\frac{dX}{X}$$

$$= \frac{2}{\sqrt{\sigma u}}L^{\frac{\nu}{\sigma}}\int_k^u (Le^{\sigma x} - K)e^{\nu x}\sin\left(\frac{\pi n x}{u}\right)dx.$$

Finally, the result (17) follows from the identity ($\omega_n = \frac{\pi n}{u}$):

$$\int_k^u e^{ax}\sin(\omega_n x)\,dx = \frac{1}{\omega_n^2 + a^2}\left[e^{ak}(\omega_n\cos(\omega_n k) - a\sin(\omega_n k)) - (-1)^n\omega_n e^{au}\right]. \ \square$$

Proof of Proposition 2. By the Feynman-Kac theorem, the function $V(x,T)$ defined by Eq.(20) for any $\mathcal{L}_2$ payoff f is a unique continuous solution of the PDE (the operator $\mathcal{A}$ is defined in Eq.(26)) $\mathcal{A}V + \frac{\partial V}{\partial T} = 0$, $x \in (L,U)$, $T \in (0,\infty)$, with absorbing boundary conditions $V(L,T) = V(U,T) = 0$, $T \in [0,\infty)$, and the initial condition $V(x,0) = f(x)$, $x \in (L,U)$. If $\varphi(x_T)$ is an eigenpayoff, then $V(x,T) = e^{-\rho T}\varphi(x)$ for some real ρ. Substituting this into the PDE we find that $\varphi(x)$ must satisfy the Sturm-Liouville ODE with absorbing boundary conditions (27). It follows from the standard facts of the Sturm-Liouville theory (Dunford and Schwartz (1963), Stakgold (1998), Zwillinger (1998)) that the spectrum of a regular Sturm-Liouville problem on the finite interval $[L,U]$ with $a(x) > 0$ on $[L,U]$, absorbing boundary conditions at both end-points of the interval, and non-negative continuous potential $R(x)$ is simple, purely discrete and positive: $0 < \rho_1 < \rho_2 < \ldots \rho_n < \ldots$ with $\rho_n \to \infty$ as $n \to \infty$. The associated eigenfunctions φ_n form a complete orthonormal basis in the Hilbert space $\mathcal{L}_2([L,U],\mathfrak{m})$ of functions on $[L,U]$ square-integrable with the weight $\mathfrak{m}$ defined in (22). Then any $\mathcal{L}_2$ payoff is in the span of φ_n, and the coefficients f_n in (25) are determined by calculating the inner products of the payoff function with the eigenpayoffs. The convergence is in the norm of the Hilbert space. This proves parts (i) and (ii). Finally, the pricing formula (28) follows from the eigenfunction expansion of the payoff (25), the linearity of the pricing operator, and the eigenvector property of φ_n (24).□

Proof of Proposition 3. The coefficients f_n are given by the inner product of the call payoff with the eigenpayoffs (60). From Eq.(63) we have:

$$f_n = (f, \varphi_n) = \int_K^U (Y-K)\varphi_n(Y)\mathfrak{m}(Y)dY = N_n \frac{\Gamma(\frac{1}{2}+m-k)}{\Gamma(1+2m)} \left[W_{k_n,m}(l)I_n - M_{k_n,m}(l)J_n\right],$$

where ($y := \frac{\mu}{\delta^2|\beta|}Y^{-2\beta}$)

$$I_n = \int_K^U (Y-K)Y^{\frac{1}{2}+\beta}e^{-\frac{y}{2}}M_{k_n,m}(y)\mathfrak{m}(Y)dY, \tag{120}$$

$$J_n = \int_K^U (Y-K)Y^{\frac{1}{2}+\beta}e^{-\frac{y}{2}}W_{k_n,m}(y)\mathfrak{m}(Y)dY. \tag{121}$$

Substituting the expression (46) for $\mathfrak{m}(Y)$ and using the indefinite integrals in Slater (1960), pp.23-25, the integrals I_n and J_n are calculated in closed form yielding Eqs.(66), (67).□

Proof of Proposition 4. The domain of the up-and-out problem is $[0, U]$. For $\beta < -\frac{1}{2}$, zero is a regular boundary point for the CEV diffusion, and we impose a killing boundary condition. The corresponding Sturm-Liouville problem on the interval $[0, U]$ with two absorbing boundary conditions at 0 and U is regular, and Proposition 2 holds in the limiting case $L = 0$ when $\beta < -\frac{1}{2}$ (see Remark 1 and note that both the scale and speed density (46) are absolutely integrable near zero). For $-\frac{1}{2} \le \beta < 0$, zero is an exit boundary for the CEV diffusion and no additional boundary condition can be imposed in this case. The corresponding Sturm-Liouville problem is singular (the speed density (46) is not integrable near zero). Thus, Proposition 2 is not immediately applicable in the limiting case $L = 0$ when $-\frac{1}{2} \le \beta < 0$, and further analysis is necessary. To analyze the nature of the spectrum, we use the results collected in Pruess, Fulton and Xie (1996) and Zwillinger (1998) p.97 (see also Dunford and Schwartz (1963)). First, using the *Liouville transformation*

$$t = -\frac{\sqrt{2}}{\delta\beta}S^{-\beta}, \;\; v(t) = f(t)u(S(t)),$$

$$f(t) = \frac{2^{\frac{1}{4}}}{\sqrt{\delta}}\left(-\frac{\delta\beta t}{\sqrt{2}}\right)^{\frac{1}{2}+\frac{1}{2\beta}} \exp\left(-\frac{\mu\beta}{4}t^2\right), \;\; S(t) = \left(-\frac{\delta\beta t}{\sqrt{2}}\right)^{-\frac{1}{\beta}},$$

we transform the CEV ODE $\mathcal{A}u = \rho u$ with $\mathcal{A}$ given by Eq.(47) into the *Liouville normal form*

$$-v_{tt} + Q(t)v = \rho v,$$

$$Q(t) = \frac{a_1}{t^2} + a_2 + a_3 t^2, \;\; a_1 = \frac{1}{4}\left(\frac{1}{\beta^2} - 1\right), \;\; a_2 = -\mu\beta\left(\beta + \frac{1}{2}\right), \;\; a_3 = \frac{\mu^2\beta^2}{4}.$$

Examining the asymptotic behavior of the *potential* $Q(t)$, we conclude that the problem with $\beta < 0$ and $\mu > 0$ is *non-oscillatory* both at $S = 0$ $(t = 0)$ and $S = \infty$ $(t = \infty)$ for all ρ. Thus, the spectra of the up-and-out problem on the interval $[0, U]$, down-and-out problem on $[L, \infty)$, and vanilla problem on $[0, \infty)$ are all simple, purely discrete, and bounded below (positive) just as in the case of the regular double knock-out problem with two barriers placed at $L > 0$ and $U < \infty$.

The eigenvalues and eigenfunctions of the up-and-out problem are found as follows. The Green's function (37) for the double knock-out problem on $[L, U]$ is $(y := \frac{\mu}{\delta^2|\beta|} Y^{-2\beta})$

$$G_\lambda^{DB}(S, Y) = \mathfrak{m}(Y)\frac{\delta^2}{2\mu}(SY)^{\frac{1}{2}+\beta} e^{-\frac{x+y}{2}} \frac{\Delta_{k,m}(l, x \wedge y)\Delta_{k,m}(x \vee y, u)}{\Delta_{k,m}(l, u)}. \quad (122)$$

The limit $L \to 0$ $(l \to 0)$ yields the Green's function G_λ^{UO} for the up-and-out problem on $[0, U]$

$$G_\lambda^{UO}(S, Y) = \mathfrak{m}(Y)\frac{\delta^2}{2\mu}(SY)^{\frac{1}{2}+\beta} e^{-\frac{x+y}{2}} \frac{M_{k,m}(x \wedge y)\Delta_{k,m}(x \vee y, u)}{M_{k,m}(u)}. \quad (123)$$

To arrive at the limit we used the representation (63) for the function $\Delta_{k,m}(a, b)$ and the asymptotic properties of the Whittaker functions (Slater (1960))

$$M_{k,m}(l) \sim l^{m+\frac{1}{2}} e^{-\frac{l}{2}} \quad \text{and} \quad W_{k,m}(l) \sim \frac{\Gamma(2m)}{\Gamma\left(m + k + \frac{1}{2}\right)} l^{-m+\frac{1}{2}} e^{-\frac{l}{2}} \quad \text{as} \quad l \to 0. \quad (124)$$

Since the spectrum of the Sturm-Liouville problem is simple, purely discrete, and positive, at a negative of an eigenvalue the Green's function G_λ^{UO} has a simple pole. Both $M_{k,m}(a)$ and $\Delta_{k,m}(a, b)$ are entire functions of k for all fixed $m > 0$ and $a, b > 0$, and therefore $M_{k,m}(u)$ in the denominator of (123) must have a simple zero at $\lambda = -\rho_n$. Thus, the zeros $\{k_n, n = 1, 2, \ldots\}$ of the Whittaker function $M_{k,m}(u)$ (considered as a function of k with both m and u fixed) produce the eigenvalues ρ_n according to the formula (58). At the point $k = k_n$

$$\Delta_{k_n,m}(a, u) = -\frac{\Gamma(\frac{1}{2} + m - k_n)}{\Gamma(1 + 2m)} M_{k_n,m}(a) W_{k_n,m}(u), \;\; a \le u,$$

and the residue is

$$\mathrm{Res}_{\lambda=-\rho_n} G_\lambda^{UO}(S, Y) = \mathfrak{m}(Y)\frac{\delta^2|\beta|\Gamma(\frac{1}{2} + m - k_n) W_{k_n,m}(u)}{\Gamma(1 + 2m)\left[\frac{\partial M_{k,m}(u)}{\partial k}\right]\Big|_{k=k_n}} (SY)^{\frac{1}{2}+\beta} e^{-\frac{x+y}{2}} M_{k_n,m}(x) M_{k_n,m}(y).$$

On the other hand, from (31) the residue of $G_\lambda^{UO}(S, Y)$ at $\lambda = -\rho_n$ is equal to $\mathfrak{m}(Y)\varphi_n(S)\varphi_n(Y)$, and we recognize that the normalized eigenfunctions are given by Eqs.(71), (72).

Finally, the up-and-out call payoff is square-integrable with the speed density on the interval $[0, U]$ and, thus, its price is given by the eigenfunction expansion (69) with the coefficients $f_n = (f, \varphi_n) = N_n I_n$, where I_n is the integral (120). It was calculated in closed form in Proposition 3 and is given by Eq.(66). □

Proof of Proposition 5. The proof is similar to the proof of Proposition 4. Take the limit $U \to \infty$ ($u \to \infty$) in the expression (122) to arrive at the Green's function for the down-and-out problem:

$$G_\lambda^{DO}(S,Y) = \mathfrak{m}(Y)\frac{\delta^2}{2\mu}(SY)^{\frac{1}{2}+\beta}e^{-\frac{x+y}{2}}\frac{\Delta_{k,m}(l, x\wedge y)W_{k,m}(x\vee y)}{W_{k,m}(l)}. \tag{125}$$

To arrive at the limit we used the representation (63) for the function $\Delta_{k,m}(a,b)$ and the asymptotic properties of the Whittaker functions (Slater (1960))

$$M_{k,m}(u) \sim \frac{\Gamma(1+2m)}{\Gamma\left(\frac{1}{2}+m-k\right)}u^{-k}e^{\frac{u}{2}} \quad \text{and} \quad W_{k,m}(u) \sim u^k e^{-\frac{u}{2}} \quad \text{as} \quad u \to \infty. \tag{126}$$

Since the spectrum of the Sturm-Liouville problem is simple, purely discrete, and positive, at a negative of an eigenvalue the Green's function G_λ^{DO} has a simple pole. Both $W_{k,m}(a)$ and $\Delta_{k,m}(a,b)$ are entire functions of k for all fixed $m > 0$ and $a, b > 0$, and therefore $W_{k,m}(l)$ in the denominator of (125) must have a simple zero at $\lambda = -\rho_n$. Thus, the zeros $\{k_n, n = 1, 2, \dots\}$ of the Whittaker function $W_{k,m}(l)$ (considered as a function of k with both m and l fixed) produce the eigenvalues ρ_n according to the formula (58). At the point $k = k_n$

$$\Delta_{k_n,m}(l,a) = -\frac{\Gamma(\frac{1}{2}+m-k_n)}{\Gamma(1+2m)}W_{k_n,m}(a)M_{k_n,m}(l), \quad a \ge l,$$

and the residue is

$$\mathrm{Res}_{\lambda=-\rho_n}G_\lambda^{DO}(S,Y) = \mathfrak{m}(Y)\frac{\delta^2|\beta|\Gamma(\frac{1}{2}+m-k_n)M_{k_n,m}(l)}{\Gamma(1+2m)\left[\frac{\partial W_{k,m}(l)}{\partial k}\right]\Big|_{k=k_n}}(SY)^{\frac{1}{2}+\beta}e^{-\frac{x+y}{2}}W_{k_n,m}(x)W_{k_n,m}(y).$$

On the other hand, from (31) the residue of $G_\lambda^{DO}(S,Y)$ at $\lambda = -\rho_n$ is equal to $\mathfrak{m}(Y)\varphi_n(S)\varphi_n(Y)$, and we recognize that the normalized eigenfunctions are given by Eqs.(76)-(77).

Finally, the down-and-out put payoff is square-integrable with the speed density on the interval $[L, \infty)$ and, thus, its price is given by the eigenfunction expansion (74) with the coefficients $f_n = (f, \varphi_n) = N_n \int_L^K (K-Y)Y^{\frac{1}{2}+\beta}e^{-\frac{y}{2}}W_{k_n,m}(y)\mathfrak{m}(Y)dY$. The integral is calculated in closed form similar to the integrals (120), (121) by using the indefinite integrals in Slater (1960), pp.23-25. □

Proof of Proposition 6. For any $\alpha \ge 0$ and $0 < L < S < \infty$, introduce the following notation:

$$\Psi_\alpha^-(T; S, L) := E_S[e^{-\alpha T_L}\mathbf{1}_{\{T_L \le T\}}]. \tag{127}$$

We need to compute the down-and-in forward price:

$$f_{DI} = e^{-rT} E_S\left[(S_T - K)\mathbf{1}_{\{T_L \le T\}}\right] = e^{-rT} E_S\left[S_T \mathbf{1}_{\{T_L \le T\}}\right] - e^{-rT} K E_S\left[\mathbf{1}_{\{T_L \le T\}}\right].$$

The expectation in the first term simplifies as follows:

$$E_S\left[S_T \mathbf{1}_{\{T_L \le T\}}\right] = E_S\left[E[S_T|\mathcal{F}_{T_L}]\mathbf{1}_{\{T_L \le T\}}\right] = E_S[e^{\mu(T-T_L)} L\, \mathbf{1}_{\{T_L \le T\}}] = e^{\mu T} L\, E_S[e^{-\mu T_L}\mathbf{1}_{\{T_L \le T\}}].$$

Then the down-and-in forward price takes the form:

$$f_{DI} = e^{-qT} L\, \Psi^-_{\mu}(T;S,L) - e^{-rT} K\, \Psi^-_0(T;S,L). \tag{128}$$

From Propositions 1, 2, and 5 in Davydov and Linetsky (2000), for any $\lambda > 0$ the Laplace transform of the function $\Psi^-_{\alpha}(T;S,L)$ in time to maturity T can be expressed in the form:

$$\int_0^{\infty} e^{-\lambda T} \Psi^-_{\alpha}(T;S,L) dT = \frac{S^{\frac{1}{2}+\beta} e^{-\frac{x}{2}} W_{m-\frac{1}{2}-\frac{\lambda+\alpha}{2\mu|\beta|},m}(x)}{\lambda L^{\frac{1}{2}+\beta} e^{-\frac{l}{2}} W_{m-\frac{1}{2}-\frac{\lambda+\alpha}{2\mu|\beta|},m}(l)}, \quad 0 < L \le S < \infty. \tag{129}$$

We invert this Laplace transform by means of the Cauchy Residue Theorem (see Doetsch (1974), pp.169-173). As a function of the complex variable λ, the right-hand side of Eq.(129) (in what follows denoted by $F(\lambda)$) is a ratio of two entire functions. The denominator has simple zeros at $\lambda = 0$ and $\lambda = -(\alpha + \rho_n)$, $n = 1, 2, \dots$, where ρ_n are the eigenvalues of the down-and-out problem. Moreover, using the asymptotic properties of the Whittaker function $W_{k,m}(a)$ when $|k| \to \infty$ (Slater (1960), p.70), we conclude that the technical conditions (Doetsch (1974), p.171, Hypotheses H1 and H2) are satisfied, and the inverse Laplace transform is expressed as a sum of residues of the function $e^{\lambda T} F(\lambda)$ at $\lambda = 0$ and $\lambda = -(\alpha + \rho_n)$, $n = 1, 2, \dots$ (Doetsch (1974), p.171, Eqs.(3), (5)):

$$\Psi^-_{\alpha}(T;S,L) = \frac{S^{\frac{1}{2}+\beta} e^{-\frac{x}{2}}}{L^{\frac{1}{2}+\beta} e^{-\frac{l}{2}}} \left\{ \frac{W_{m-\frac{1}{2}-\frac{\alpha}{2\mu|\beta|},m}(x)}{W_{m-\frac{1}{2}-\frac{\alpha}{2\mu|\beta|},m}(l)} + \sum_{n=1}^{\infty} \frac{2\mu|\beta| e^{-(\alpha+\rho_n)T} W_{k_n,m}(x)}{(\alpha+\rho_n)\left[\frac{\partial W_{k,m}(l)}{\partial k}\right]\Big|_{k=k_n}} \right\},$$

where k_n are the roots of Eq.(75). Substituting this result into Eq.(128) and observing that (see Abramowitz and Stegun (1972), p.510; $\Gamma(\nu,x) = \Gamma(\nu)G(\nu,x)$ is the incomplete Gamma function)

$$W_{m-\frac{1}{2},m}(x) = x^{-m+\frac{1}{2}} e^{\frac{x}{2}} \Gamma(2m,x), \quad W_{-m-\frac{1}{2},m}(x) = x^{m+\frac{1}{2}} e^{\frac{x}{2}} \Gamma(-2m,x),$$

after some algebra we arrive at the final result (81) for the present value of the down-and-in forward contract. □

Proof of Proposition 7. The proof is similar to the proof of Proposition 6. From Propositions 1, 2, and 5 in Davydov and Linetsky (2000), for any $\lambda > 0$ the Laplace transform of $E_S[e^{-rT_U}\mathbf{1}_{\{T_U \le T\}}]$ in time to maturity T is:

$$\int_0^{\infty} e^{-\lambda T} E_S[e^{-rT_U}\mathbf{1}_{\{T_U \le T\}}] dT = \frac{S^{\frac{1}{2}+\beta} e^{-\frac{x}{2}} M_{m-\frac{1}{2}-\frac{\lambda+r}{2\mu|\beta|},m}(x)}{\lambda U^{\frac{1}{2}+\beta} e^{-\frac{u}{2}} M_{m-\frac{1}{2}-\frac{\lambda+r}{2\mu|\beta|},m}(u)}, \quad 0 < S \le U < \infty. \tag{130}$$

We invert this Laplace transform similar to (129) by means of the Cauchy Residue Theorem. As a function of λ, the right-hand side of Eq.(130) is a ratio of two entire functions. The denominator has simple zeros at $\lambda = 0$ and $\lambda = -(r + \rho_n)$, $n = 1, 2, \ldots$, where ρ_n are the eigenvalues of the up-and-out problem. Moreover, using the asymptotic properties of the Whittaker function $M_{k,m}(a)$ when $|k| \to \infty$ (Slater (1960), p.70), we conclude that the technical conditions are satisfied, and the inverse Laplace transform is expressed as a sum of residues at $\lambda = 0$ and $\lambda = -(r + \rho_n)$, $n = 1, 2, \ldots$, yielding the result (83).□

Proof of Proposition 8. (i) As discussed in the proof of Proposition 4, the spectrum of the singular Sturm-Liouville problem on $[0, \infty)$ is simple, purely discrete, and bounded below. Take the limit $L \to 0$ ($l \to 0$) in the expression (129) to arrive at the Green's function for the problem on $[0, \infty)$:

$$G_\lambda(S, Y) = \mathfrak{m}(Y) \frac{\delta^2}{2\mu} (SY)^{\frac{1}{2}+\beta} e^{-\frac{x+y}{2}} \frac{\Gamma(\frac{1}{2} + m - k)}{\Gamma(1 + 2m)} M_{k,m}(x \wedge y) W_{k,m}(x \vee y). \tag{131}$$

To arrive at the limit we used the representation (63) for $\Delta_{k,m}(a, b)$ and the fact that $M_{k,m}(0) = 0$ for all complex k and $m > 0$. Since the spectrum of the Sturm-Liouville problem is simple and purely discrete, at a negative of an eigenvalue the Green's function G_λ has a simple pole. Both $W_{k,m}(a)$ and $M_{k,m}(a)$ are entire functions of k for all fixed $m > 0$ and $a > 0$. The Gamma function $\Gamma(\frac{1}{2} + m - k) = \Gamma(\frac{\lambda}{2\mu|\beta|} + 1)$ in the numerator of (125) has simple poles at $\lambda = -2\mu|\beta|n$, $n = 1, 2, \ldots$, with the residues $\frac{(-1)^{n-1} 2\mu|\beta|}{(n-1)!}$. Thus the eigenvalues are $\rho_n = 2\mu|\beta|n$, $n = 1, 2, \ldots$. The residues of the Green's function at $\lambda = -\rho_n$ are

$$\text{Res}_{\lambda=-\rho_n} G_\lambda(S, Y) = \mathfrak{m}(Y) \frac{(-1)^{n-1}\delta^2|\beta|(SY)^{\frac{1}{2}+\beta} e^{-\frac{x+y}{2}}}{(n-1)!\Gamma(1 + 2m)} M_{m+n-\frac{1}{2},m}(x \wedge y) W_{m+n-\frac{1}{2},m}(x \vee y).$$

When $k = m + n - \frac{1}{2}$, $n = 1, 2, \ldots$, the functions $M_{k,m}(x)$ and $W_{k,m}(x)$ become linearly dependent and reduce to generalized Laguerre polynomials (Abramowitz and Stegun (1972), pp.505 and 509-10)

$$M_{m+n-\frac{1}{2},m}(x) = \frac{(n-1)!\Gamma(1 + 2m)}{\Gamma(2m + n)} e^{-\frac{x}{2}} x^{m+\frac{1}{2}} L_{n-1}^{2m}(x),$$

$$W_{m+n-\frac{1}{2},m}(x) = (-1)^{n-1}(n-1)! e^{-\frac{x}{2}} x^{m+\frac{1}{2}} L_{n-1}^{2m}(x).$$

Then the residues can be re-written in the form

$$\text{Res}_{\lambda=-\rho_n} G_\lambda(S, Y) = \mathfrak{m}(Y) \frac{\delta^2|\beta|(n-1)!}{\Gamma(n + 2m)} \left(\frac{\mu}{\delta^2|\beta|}\right)^{1+2m} (SY) e^{-\frac{x+y}{2}} L_{n-1}^{2m}(x) L_{n-1}^{2m}(y).$$

On the other hand, $\text{Res}_{\lambda=-\rho_n} G_\lambda(S, Y) = \mathfrak{m}(Y)\varphi_n(S)\varphi_n(Y)$ and we recognize the eigenfunctions (85).

(ii) The continuous density $p(T; S, S_T)$ is defective. Integrating the representation (90) produces the probability of absorption (87) via the relationship: $\int_0^\infty p(T; S, S_T) dS_T = 1 - \Pr(S_T = 0 | S_0 = S)$.

(iii) Similar to the down-and-out call, the vanilla call is not in $\mathcal{L}_2([0,\infty), \mathfrak{m})$. We price the vanilla put first. To price the put, we decompose the put payoff into two parts: $(K - S_T)^+ = K\mathbf{1}_{\{T_0 < T\}} + (K - S_T)^+ \mathbf{1}_{\{T_0 \geq T\}}$. The first part is the "bankruptcy claim" that pays off the strike price K in the case of absorption at zero (bankruptcy) prior to and including maturity T. The price of the bankruptcy claim contributes the first term in (87). The second part can be interpreted as a down-and-out put with the barrier placed at zero. Its terminal payoff is in $\mathcal{L}_2([0,\infty), \mathfrak{m})$ and its price is given by the eigenfunction expansion in (87). The coefficients of the expansion (88) are calculated in closed form using the integrals in Prudnikov, Brychkov, and Marichev (1986), Vol. 2, pp.51 and 463. Finally, the vanilla call price is recovered from the put-call parity.□

Proof of Proposition 9. (i) The proof is similar to (i) of Proposition 8. (ii) The zero-coupon bond payoff $f(r_T) = 1$ is in the span of the eigensecurities ($f = 1$ is square-integrable on $(0, \infty)$ with the weight (98)). The coefficients $f_n = (1, \varphi_n)$ are calculated in closed form using the integral (Gradshteyn and Ryzhik (1994), p.850) (for all $\alpha > -1$, $s > 0$, $n = 0, 1, 2, ...$)

$$\int_0^\infty e^{-sx} x^\alpha L_n^\alpha(x) dx = \frac{\Gamma(\alpha + n + 1)(s-1)^n}{n!\, s^{\alpha+n+1}}.\square$$

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