Equivalent martingale measures and Lévy's theorem

David Heath
Carnegie Mellon University

The work presented in this talk is from the Ph.D. thesis of Diego Jara

Derivatives pricing

- Based on the "Fundamental Theorem of Finance":
  - Either there is arbitrage or there is an equivalent martingale measure (roughly speaking)
- Equivalence: on $(\Omega, \mathcal{F})$, P is equivalent to Q ⇔
  - $(P(A)=1 \Leftrightarrow Q(A)=1$ for all A in $\mathcal{F})$ or
  - $P \ll Q$ and $Q \ll P$
- All we need to know is the equivalence class of the "physical" probability

Identifying the equivalence class of P

- Let X be the price of any security expressed in terms of a numeraire security (so 1 is a security price process).
- For there to be an e.m.m., X must be a semimartingale
- $\lim_{m \to \infty} X(0)^2 + \sum_{i=1}^{n_m} (X(t_i^{(m)}) - X(t_{i-1}^{(m)}))^2 = [X, X]_t$ w. prob. 1 for suitable sequences of partitions

A necessary condition for equivalence

- We must correctly model the quadratic variation process
- Sometimes that's enough:
  - Suppose $[X,X]_t=t$ for all t. Then X is continuous
  - If X is also a local martingale, then Lévy's theorem says that X is a standard Brownian motion.
- Question: When is knowing the law of the quadratic variation process sufficient for obtaining the law of a martingale?

Not always!

- Consider X defined by $X_t = \int_0^t W_s dW_s = \frac{W_t^2 - t}{2}.$
  - (W is standard Brownian motion).
- Clearly $[X,X]_t = \int_0^t W_s^2 ds$
- But: $[-X,-X]_t = [X,X]_t$ and X and -X have different laws!

Which martingales are characterized by their quadratic variation?

- Let *M* be the class of continuous local martingales starting at 0.
- Definition: M *M* is called divergent if $[M,M]_{\infty} = \infty$ a.s. Let *D* be the class of divergent M *M*.
- Theorem 1. Let M *D* have absolutely continuous quadratic variation with $d[M,M]_t/dt > 0$. Then

(*) (N *M*, $[N,N] =^d [M,M]$ implies $N =^d M$) =>

M is a Gaussian process

Proof of Theorem 1

- Suppose M is a divergent continuous local martingale, and that M is not Gaussian.
- Dambanis and Dubins-Schwartz (DDS) showed that $M_t=B_{[M,M](t)}$ for some Brownian motion B.
- If [M,M] were a deterministic process, then (from this representation) M would be Gaussian.
- Hence d[M,M](t)/dt is not a deterministic process

- We now construct (adapted) processes W and Y on some (other) filtered space (Ω, F, P), (F_t), with W a Brownian motion and Y having the same distribution as X where $X_t = \sqrt{d[M,M](t)/dt}$ and W and Y are not independent.
- Define the martingale N by N=Y · W.
- By our construction we have $[N,N] =^d [M,M]$
- We need to show that N and M do not have the same distribution. And this would violate condition (*) of the theorem.

A result of Vostrikova and Yor (1999)

- Definition A continuous local martingale is called Ocone if, in its DDS decomposition, B and [M,M] are independent.
- Ocone showed: A continuous local martingale X is Ocone ⇔ $X =^d e \cdot X$ for every predictable process e satisfying $|e|=1$.
- An M satisfying condition (*) of our theorem would automatically satisfy this weaker condition. Hence M is Ocone.

Vostrikova and Yor showed:

- Let $\{W_t, F_t\}$ be a Brownian motion and $\{Y_t\}$ be strictly positive, F_t-adapted with $\overset{T}{\underset{0}{}} Y_s^2 ds < \times \quad t < \times$ and $\overset{\times}{\underset{0}{}} Y_s^2 ds = \times$
- Then N=Y·W is Ocone iff W and Y are independent, and we purposely chose them not to be independent. Hence N is not Ocone. So N and M cannot have the same distribution.

The above result was negative

- It says: If the distribution of a martingale is characterized by that of its [,] process, then the martingale must be Gaussian.

- For a more positive result …

Suppose we consider diffusions:

- For any real Borel measurable function f, define Z(f)=the set of x such that f(x)=0, and define I(f) = the set of x such that (1/f) is not locally square integrable at x.

- Example: If f has a continuous first derivative everywhere then I(f)=Z(f).

Theorem 2

- Suppose g_1 and g_2 are Borel measurable functions on R, and $I(g_i)=Z(g_i)$ for i=1, 2. Let $(X^{(i)},W^{(i)})$, $(\Omega^{(i)},F^{(i)},P^{(i)})$ be weak solutions to the equations $X_0=0$, $dX_t=g_i(X_t)dW_t$ for i=1,2.

If $[X^{(1)},X^{(1)}] =^d [X^{(2)},X^{(2)}]$ then either $X^{(1)} =^d X^{(2)}$ or $X^{(1)} =^d -X^{(2)}$.

Proof of Theorem 2

- Not trivial. Uses:
  - Reduction to the case $g_i \geq 0$.
  - Showing that the quadratic variation of solutions being the same implies g_1 is essentially equal to g_2.
  - A result of Engelbert and Schmidt (1984):
    For every initial distribution μ, the equation $dX_t = g_1(X_t)dW_t$ has a solution which is unique in the sense of probability law if and only if $I(g_1) = Z(g_1)$.

A final example

- Take $g_1(x)=|x|+1$
- Let (X,W) be a weak solution to $dX_t=g_1(X_t)dW_t$ with $X_0=0$.
- Clearly $Z(g_1)=I(g_1)=$.
- Then $d(-X_t)=-g_1(X_t)dW_t=g_1(-X_t)(-dW_t)$ so the (Y,B), with Y=-X, B=-W, is also a weak solution. Since weak uniqueness holds, the distribution of X and Y=-X must be the same.

- Theorem 2 tells us: this X is the only solution to $dY_t=g(Y_t)dW_t$ with $Y_0=0$ (for any g which stays away from 0) having quadratic variation process equal to that of X. (no +/- problems; symmetric)
- But X is not Gaussian. Hence (by Theorem 1) there must be some other martingale starting at 0 with the same quadratic variation as X.

Here are some:

- Choose an a>0
- Define Y by:

$$Y_t = \begin{cases} X_t \text{ if } t \le a \\ 2X_a - X_t \text{ if } t \,?\, a \end{cases}$$

- Y is a continuous martingale, [Y,Y]=[X,X] and Y_0=0.

If you're interested in a copy of Diego's thesis

- You could contact Diego (is he here?)
- You could write to me:
  heath@andrew.cmu.edu
  and I'll send a pdf file vie email