Pricing and Risk-Managing Exotics On Assets with Stochastic Volatility

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Outline of the Presentation

1. Main Objectives:

Development of new methods for pricing and risk management of financial instruments in the post-Black-Scholes framework.

2. Typical Derivative Securities and their Conceptual Pricing:

Plain vanilla options, European options with general payoffs, weakly path-dependent options (American, barrier, lookback), strongly path-dependent options (Asian, timers, faders, options on volatility, etc.), options on the trading account.

3. Deviations from Black-Scholes Paradigm and their Implications:

Short selling constraints, liquidity, transaction costs, discrete hedging.

4. Options on Trading Account and P&L Distributions:

Passport options, vacation options, swing options, distributions of the P&L, examples.

5. Calls and Bets:
To hedge or not to hedge in case of parameter misspecification.

6. Enigma of Smile and its Possible Solutions:
Implied volatility, local volatility, complementary inverse problems, sticky strike and sticky delta models, jumps, random mixing of lognormal distributions, stochastic volatility, combinations of the above.

7. Exact Pricing of Exotic Options on Assets with Stochastic Volatility:
Options on volatility, barrier options, lookback options, passport options, forward starting options, etc.

8. Robust Hedging and Practicalities

9. Conclusions

Detailed Exposition

A. Lipton, Mathematical Methods for Foreign Exchange (A Financial Engineer's Approach), to be published shortly by World Scientific Publishing Co.

Useful References

Andersen, Andreason & Brotherton-Radcliffe (1998), Andreason (1998), Andersen, Andreason (2000), Avellaneda & Paras (1994), Black & Scholes (1973), Britten-Jones & Neuberger (2000), Carr (1990-2000), Delbaen & Yor (1999), Derman (1996), Derman & Kani (1994), Dupire (1994, 1996), Esipov & Vaysbourd (1999), Henderson & Hobson (1999, 2000), Heston (1993), Hull & White (1987), Hyer, Lipton, Pugachevsky (1997), Inglis & Lipton (1998), Jaillet & Ronn (1998), Jarrow & Rudd (1982), Kocic (1997), Krakovsky (1999), Leland (1985), Linetsky & Davidov (2000), Lipton (1996-2000), Lipton & Little (2000), Lipton & McGhee (1999), Lipton & Pugachevsky (1998), Matytsin (1999), Merton (1973), Rubinstein (1994), Samuelson (1965), Schmock, Schreve & Wystup (1999), Scott (1987), Shreve & Vecer (1998, 2000), Schweizer (1992), Stein & Stein (1991), Wilmott, Howison, Dewynne (1995), Wilmott (1998), etc.

Black-Scholes Equation for Pricing European Calls

Early precursor – probabilistic valuation (Bonnes 1964).

The magic of hedging: (Black & Scholes 1973):

$$C(t,S,T,K) = SN(d_1) - e^{-r(T-t)}KN(d_2),$$

$$d_{1,2} = \frac{\ln(S/K) + (r \pm s^2/2)(T-t)}{s\sqrt{T-t}}.$$

Alternative view: risk-neutral valuation (Harrison & Pliska 1981, others).

Yet another approach: homogeneity (Merton 1974, Kocic 1997, others).

Black-Scholes formula is very powerful but it cannot be used in its original form for practical purposes. Accordingly, there is a need for the development of new methods for pricing and risk management of financial instruments in the post-Black-Scholes framework

Typical Derivative Securities and their Pricing:

Plain vanilla options, European options with general payoffs, weakly path-dependent options (American, barrier, lookback), strongly path-dependent options (Asian, timers, faders, options on volatility, etc.), options on the trading account.

The pricing formula based on risk-neutral valuation reads:

$$payoff = \Phi\left(S_0, S_{t_1}, \ldots, S_{t_{N-1}}, S_T\right),$$

$$\begin{aligned} price = e^{-r^0 T} \iiint \Phi\left(S_0, S_{t_1}, \ldots, S_{t_{N-1}}, S_T\right) \\ \times p\left(0, S_0, t_1, S_{t_1}\right) \ldots p\left(t_{N-1}, S_{t_{N-1}}, T, S_T\right) dS_{t_1} \ldots dS_T, \end{aligned}$$

or, more generally,

$$price = e^{-r^0 T} \int_{\Omega} \Phi(w) dD(w).$$

In the Black-Scholes framework the t.p.d.f.'s are log-normal, however, in reality they are far from being log-normal.

Deviations from Black-Scholes Paradigm and their Implications:

In actual markets the key Black-Scholes assumptions are routinely violated. In many cases these violations can be accounted for by judicious alternations of the original pricing problem. We consider a few representative situations.

1. There is no reason to believe that the (risk-neutralized) dynamics of the underlying price is log-normal, i.e.,

$$\frac{dS_t}{S_t} = \left(r^0 - r^1\right)dt + s\,dW_t.$$

In reality it can be much more complex (a more general diffusion, a process with stochastic volatility, a Levy process, etc.).

2. The standard price of the up-and-out call is governed by the non-dimensional pricing problem of the form:

$$U_t - \tfrac{1}{2}U_{XX} = 0, \qquad U(0,X) = \left(e^{X/2} - e^{-X/2}\right)_+, \quad U(t,B) = 0.$$

Solution of this problem results in huge Delta and Gamma near the barrier which can easily violate the natural short-selling constraints. The familiar trader's trick is to move the barrier upward. Normally the barrier is kept straight. Occasionally, it is made curvilinear. It is also possible to replace the Dirichlet boundary condition at $X=B$ with the mixed boundary condition of the form (Schmock et al., 1999):

$$U_X(t,B) + bU(t,B) = 0, \qquad b > 0.$$

The best way to solve the modified problem is to introduce the new variable

$$W(t,X) = U_X(t,X) + bU(t,X),$$

which satisfies the same problem as *U* but with the standard Dirichlet boundary condition. It is easy to express *U* in terms of *W* by integration with respect to *X*:

$$U(t,X) = \int_0^X e^{b(X'-X)} W(t,X') dX'.$$

This trick is useful in many situations, for instance for pricing lookback and passport options (Lipton 2000).

3. To address the short-selling constraint, we do not need to change the Black-Scholes equation. This is not the case if we want to take into account liquidity effects, i.e., the impact of hedging of large books of options on the underlying. In the simplest case, the pricing equation assumes the form (Jarrow, Krakovsky 1999, and others):

$$V_t(t,S) + \tfrac{1}{2} \frac{s^2 S^2}{\left(1 + V_{SS}(t,S)/L\right)^2} V_{SS}(t,S) + rSV_S(t,S) - rV(t,S) = 0.$$

Here *L* is a dimensional parameter characterizing the liquidity of the underlying. Although this equation is highly nonlinear, it is relatively easy to solve numerically. It is clear that it can be used to analyze the short-selling constraint.

4. One more effect which is of great importance for practitioners is the impact of transaction costs on option prices especially because these costs make continuous hedging impossible. Denoting by *k* the proportional round-trip transaction costs, we can write the pricing problem in the presence of transaction costs as (Leland, 1985, Avellaneda & Paras 1994, and Soner *et al*. 1995, and others):

$$V_t(t,S) + \tfrac{1}{2}\sigma^2\left(1 + A\,\mathrm{sign}\left(V_{SS}(t,S)\right)\right)V_{SS}(t,S) + rSV_S(t,S) - rV(t,S) = 0.$$

Here $A = \sqrt{\frac{2}{\pi}}\left(\frac{k}{\sigma\sqrt{\delta t}}\right)$ is the celebrated Leland number, and δt is the time-interval between Delta adjustments. For product with convex payoffs (for example for the European call) the pricing equation is the standard Black-Scholes equation with adjusted volatility $\hat{\sigma} = \sigma(1+A)$. Since $\hat{\sigma} \to \infty$ when $\delta t \to 0$, the price of the call in the presence of transaction costs is trivial, $C(t,S) = S$. (A more rigorous argument is presented by Soner *et al.* 1995) This is a disturbing fact which has not been addressed in full yet.

5. Even if transaction costs are absent but hedging is discrete, the standard Black-Scholes equation has to be changed. The distribution of P&L is no longer deterministic, its first and second moments are described by Bertsimas, Kogan & Lo (1998), Esipov & Vaysburd (1999), and others.

Options on Trading Account

Now we consider passport and other options on the trading account which are in a class of their own. We establish connections between them and other path dependent options. The option on the trading account gives the buyer the right to choose their own strategy (within certain limits) and keep all the gains generated by this strategy while transferring all the losses to the seller.

Let $Q_t, -a < |Q_t| < b,$ be the strategy function of the buyer and Π_t be the value of the corresponding trading account. Broadly speaking, the passport option is the zero strike option on the trading account. The governing system of SDEs for S_t, Π_t is degenerate:

$$\frac{dS_t}{S_t} = \left(r^0 - r^1\right)dt + s\,dW_t, \qquad d\Pi_t = r^0 \Pi_t dt + Q_t s\, S_t dW_t.$$

It is possible to construct at least two almost identical backward equations associated with these SDEs. The first one is the usual pricing equation for options on the trading account as originally proposed by Hyer, Lipton & Pugachevsky (1997):

$$V_t + \tfrac{1}{2} s^{\,2} S^2 \max_{-a \le Q \le b}\left(V_{SS} + 2QV_{S\Pi} + Q^2 V_{\Pi\Pi}\right) + \left(r^0 - r^1\right)SV_S + r^0 \Pi V_\Pi - r^0 V = 0.$$

As usual, this equation is augmented with the final condition

$$V\left(T, S, \Pi\right) = \Pi_+.$$

The second backward equation has the form (Esipov & Vaysburd 1999):

$$P_t + \tfrac{1}{2} s^2 S^2 \left(P_{SS} + 2QP_{S\Pi} + Q^2 P_{\Pi\Pi}\right) + \left(r^0 - r^1\right) S P_S + r^0 \Pi P_\Pi + r^0 P = 0.$$

The corresponding final condition is

$$P(T, S, \Pi) = d\left(\Pi - (S - K)_+\right).$$

This problem describes the distribution of P&L with *P* being the probability density function. Note that killing turns into creation due to dimensional requirements.

The above equations are closely connected with the portfolio optimization problem.

For a given strategy the value of the passport options can be written in the self-similar form:

$$V_Q\left(t, S, \Pi, T, r^0, r^1, s\right) = S \Phi_Q^{PO}\left(x, t, J^0, J^1\right), \qquad x = \frac{\Pi}{S}, \qquad -\infty < x < \infty.$$

For simplicity we assume that $J^1 = 0$.

In the symmetric case the choice of the optimal strategy is simple

$$Q_t = -sign(x).$$

More general optimal strategy is

$$q_t = a - sign(x - a)\, b,$$

where $a = (b - a)/2$, $b = (b + a)/2$.

Accordingly, the seller has to solve the following pricing problem for $-\infty < x < \infty$:

$$\Phi_t - \tfrac{1}{2}\left(|x-a|+b\right)^2 \Phi_{xx}, = 0,$$
$$\Phi(0,x) = x_+,$$
$$\Phi(t, x \to -\infty) = 0, \quad \Phi_x(t, x \to \infty) = 1.$$

This problem is relatively hard to solve, however, it can be greatly simplified by splitting which is a consequence of the fact that the pricing equation is invariant with respect to the discrete parity transformation $x \to -x$.

It is remarkable that this problem can be solved by splitting. The corresponding formula is

$$\Phi(t,x) = \begin{cases} x + \dfrac{a+b}{2}\Phi^{PLBP}\left(t, \dfrac{x-a+b}{b}\right) - \dfrac{1}{2}\Phi^{DOC}(t, x-a+b), & x \ge a \\ x + \dfrac{a+b}{2}\Phi^{PLBP}\left(t, \dfrac{-x+a+b}{b}\right) + \dfrac{1}{2}\Phi^{DOC}(t, -x+a+b), & x < a \end{cases}.$$

Thus, we can use results by Contze & Viswanathan (1991) to price passport. The same answer was obtained by Shreve & Vecer (1999) via a very complicated probabilistic method.

Explicit expressions are as follows:

$$\Phi(t,x)=x+\frac{b\left(1-\sqrt{t}P_-\right)}{2}N(-P_-)-\frac{(a+b)(x-a+b)}{2b}N(-P_+)+\frac{b\sqrt{t}}{2}n(P_-),$$

for $x \geq a$,

$$\Phi(t,x)=x-\frac{b\left(1+\sqrt{t}Q_{-,-}\right)}{2}N\left(-Q_{-,-}\right)+\frac{(a+b)(-x+a+b)}{2b}N\left(-Q_{-,+}\right)+\frac{b\sqrt{t}}{2}n\left(Q_{-,-}\right)$$
$$+(-x+a+b)N\left(Q_{+,+}\right)-(a+b)N\left(Q_{+,-}\right),$$

for $x < a$. Here

$$P_{\pm}=\frac{1}{\sqrt{t}}\ln\left(\frac{x-a+b}{b}\right)-\frac{1}{\sqrt{t}}\ln\left(\frac{b}{a+b}\right)\pm\frac{\sqrt{t}}{2},$$

$$Q_{\pm,\pm}=\frac{1}{\sqrt{t}}\ln\left(\frac{x-a+b}{b}\right)\pm\frac{1}{\sqrt{t}}\ln\left(\frac{b}{a+b}\right)\pm\frac{\sqrt{t}}{2}.$$

Bets – to Hedge or not to Hedge

We are interested in the case of parameter misspecification when we hedge an option with the general payoff $payoff = v(S)$, on an asset with volatility s , by using the following hedging strategy

$$q(t,S) = Q_S(t,S), \quad Q(t,S) = V^{\Sigma}(t,S),$$

where $V^{\Sigma}(t,S)$ is the price of the same option on asset with volatility Σ.

For simplicity we assume that interest rates are zero.

We introduce the adjusted P&L $p_t = \Pi_t - V_S^{\Sigma}(t,S_t)$, and write the equation for $P(t,S,p)$ as

$$P_t + \frac{1}{2}(\Sigma^2 - s^2)S^2 V_{SS}^{\Sigma} P_p + \frac{1}{2}s^2 S^2 P_{SS} = 0,$$
$$P(T,S,p) = d(p).$$

It is relatively difficult (but not impossible) to solve this backward equation. Instead, we solve the problem for the first two central moments. (A similar approach is used by Gallus 1997, Esipov & Vaysbourd 1998, and others). The non-central and central moments are

$$M^{(1)} = \int P(t,S,p)p\,dp, \quad M^{(2)} = \int P(t,S,p)p^2 dp, \quad m^{(1)} = M^{(1)}, \quad m^{(2)} = M^{(2)} - \left(M^{(1)}\right)^2.$$

Equations for $m^{(1)}, m^{(2)}$ are

$$L^{(s)} m^{(1)}(t,S) = \frac{1}{2}\left(\Sigma^2 - s^2\right) S^2 V_{SS}^{\Sigma}(t,S), \quad m^{(1)}(T,S) = 0,$$

$$L^{(s)} m^{(2)}(t,S) = -s^2 \left(S m_S^{(1)}(t,S)\right)^2, \quad m^{(2)}(T,S) = 0.$$

We can find $m^{(1)}$ explicitly,

$$m^{(1)}(t,S) = V^{(s)}(t,S) - V^{(\Sigma)}(t,S),$$

and represent $m^{(2)}$ via Duhamel's principle,

$$m^{(2)}(t,S) = s^2 \int_t^T \int_0^\infty \left(V^{(s)}(t',S') - V^{(\Sigma)}(t',S')\right)^2 p^{(s)}(t,S,t',S') S'^2 \, dt' \, dS'.$$

We are interested in hedging of a bet option so that

$$V^{(s,\Sigma)}(t,S) = N\left(d_-^{(s,\Sigma)}\right),$$

and Duhamel's integral can be found semi-explicitly. A simple algebra shows that, in principle, a mishedge can have a profound impact to the distribution of the P&L and result in a blow-up. However, in practice it does not happen. Blow-up never happens when options with continuous payoffs, such as calls and puts, are hedged.

Complementary Angle: Optimization Problem

Objective: choose an investment strategy which generates expected return on a portfolio consisting of stock, bond, and option, and has the lowest possible variance, say. (Private communications with Esipov & Vaysburd on this topic are acknowledged.)

We assume that interest rates are zero. Governing SDE’s are

$$dS_t = S_t\left(m\,dt + s\,dW_t\right), \qquad d\Pi_t = v_t\,\Pi_t\left(m\,dt + s\,dW_t\right).$$

Here $m \neq 0$ is the real-world drift. The objective is to choose v in such a way that

$$E\{\Pi_T - v(S_T)\} = p > \Pi_0, \qquad E\{\frac{1}{2}\left(\Pi_T - v(S_T)^2\right) \to \min .$$

As always, we use Lagrange multipliers and seek to minimize

$$E\{\frac{1}{2}\left(\left(\Pi_T - v(S_T)\right)^2 - l\left(\Pi_T - v(S_T)\right)\right) \to \min .$$

The corresponding Hamilton-Jacobi-Bellman (HJB) equation for $J(t, S, \Pi)$ reads

$$J_t + \min\ \{\frac{1}{2} s^2 S^2 J_{SS} + s^2 \varpi S \Pi J_{S\Pi} + \frac{1}{2} s^2 \varpi^2 \Pi^2 J_{\Pi\Pi} + m S J_S + m \varpi \Pi J_\Pi\} = 0,$$

$$J(T,S,\Pi)=\frac{1}{2}\left((\Pi-v(S))^2-l(\Pi-v(S))\right).$$

A simple calculation yields

$$\varpi=-\frac{mJ_{\Pi}+s^2SJ_{S\Pi}}{s^2\Pi J_{\Pi\Pi}}.$$

$$J_t+\frac{1}{2}s^2S^2J_{SS}+mSJ_S-\frac{(mJ_{\Pi}+s^2SJ_{S\Pi})^2}{2s^2J_{\Pi\Pi}}=0.$$

The problem can be solved via the ansatz

$$J(t,S,\Pi)=a(t,S)\Pi^2+b(t,S)\Pi+g(t,S).$$

The corresponding solution is

$$J(t,S,\Pi)=\frac{1}{2}e^{-m^2(T-t)/s^2}(\Pi-v(t,S)-l)^2-\frac{l^2}{2},$$

where $v(t,S)$ is the Black-Scholes price of the option. Alternatively, the problem can be linearized via the Legendre transform.

The distribution of $\Theta = \Pi - v(t, S)$ is shifted log-normal. It has limited upside and unlimited downside. The Sharpe ratio is

$$SR = \sqrt{e^{m^2 T / s^2} - 1}.$$

The dynamic asset allocation strategy has to be contrasted with the static asset allocation strategy which also has a log-normal with limited downside but unlimited upside and the Sharpe ratio of the form

$$SR = \frac{|1 - e^{-mT}|}{\sqrt{e^{s^2 T} - 1}}.$$

Enigma of Smile

The volatility smile effect (i.e. the need to use different volatilities to reproduce market prices of European calls and puts) is persistent in most of the markets. It is of great interest to practitioners and academics alike. Over the last decade or so many models were developed in order to describe the smile effect. For instance, Dupire (1994), Derman & Kani (1994), and Rubinstein (1994) developed the static smile model. Practitioners proposed sticky-strike and sticky-delta models (the latter one is consistent with the scaling constraints). Merton (1973), Anderson & Andreassen (1999), and many others developed jump-diffusion models. Hull & White (1987), Brigo & Mercurio (2000), and others introduced random mixtures of log-normal distributions. More general stochastic volatility models were developed by Hull & White (1987), Scott (1987), Wiggins (1987), Stein & Stein (1991), Heston (1993), Lewis (1999), and others. Various combinations of the above were proposed, for example, the universal volatility model of Dupire (1996). None of these models is entirely satisfactory from a practical viewpoint. Finding a proper theoretical framework and implementing it in practice remains a major challenge. Pricing of options in the presence of smile is very difficult very seldom can be done analytically. Asymptotic methods developed by Hull & White (1987), Hagan & Woodward (1998), Lipton (1997, 2000), and others proved to be very useful for solving the pricing problem. Needless to say that numerical methods are equally useful.

Static Smile via Method Of Reductions

Consider the Black-Scholes equation in the presence of deterministic smile:

$$V_t + \tfrac{1}{2} s^{2}(t,S) S^2 V_{SS} + \left(r^0 - r^1\right) S V_S - r^0 V = 0.$$

In the more general context we need to study equations of the form

$$V_t + \tfrac{1}{2} A(t,X) V_{XX} + B(t,X) V_X - C(t,X) V = 0,$$

where A, B, C are given functions.

It is natural to ask if it is possible to reduce the above equation either to the backward heat equation

$$U_V(V,X) + \tfrac{1}{2} U_{XX}(V,X) = 0,$$

or, more generally, to the radial heat (Lommel-Bessel) equation

$$U_V(V,X) + \tfrac{1}{2}\left(U_{XX}(V,X) + \frac{g-1}{X} U_X(V,X) \right) = 0,$$

via the following transformations

$$V = \Phi(t), \quad X = \Psi(t,S), \quad U(V,X) = e^{-\Xi(t,S)} V(t,S).$$

The answer is positive provided that the following solvability condition is satisfied

$$\Omega_t + \left(\tfrac{1}{2} A\left(\Omega_S - \Omega^2\right) + B\Omega + C\right)_S = 0,$$

$$\Omega = \frac{P_t I}{P\sqrt{A}} + \frac{I_t}{\sqrt{A}} + \frac{Q_t}{P\sqrt{A}} - \frac{A_S}{4A} + \frac{B}{A} + \frac{(1-g)P}{2(PI+Q)\sqrt{A}},$$

$$P = P(t), \quad Q = Q(t), \quad I = \int \frac{dS}{\sqrt{A(t,S)}}.$$

The reduction method is very powerful and provides a large number of interesting examples. For instance, it can be used in order to price options on banded forex and options on forex with constant elasticity of variance.

Example: Banded Forex Model via Reductions

For banded forex the pricing problem written in terms of the forward price as a function of the forward rate has the form

$$V_t + \tfrac{1}{2}\frac{s^2(F-L)^2(U-F)^2}{(U-L)^2}V_{FF} = 0,$$
$$V(T,F) = (F-K)_+.$$

It can be reduced to the final value problem for the heat equation and priced explicitly (Ingersoll 1996, Rady 1997). The answer is

$$V(t,F) = \frac{(U-K)(F-L)}{(U-L)}N(d_+(t,F)) - \frac{(K-L)(U-F)}{(U-L)}N(d_-(t,F)),$$
$$d_\pm = \frac{\ln((U-K)(F-L)/(K-L)(U-F))}{s\sqrt{T-t}} \pm \frac{s\sqrt{T-t}}{2}.$$

Example: CEV Model via Reductions

The CEV assumption $s\,(t,S) = s\,(t)S^{p}$. Natural restrictions on the power: $0 \le p \le 2$ since this interval is invariant with respect to the transform $p \to 2-p$ which describes the switch between the domestic and foreign countries. The CEV pricing equation has the form

$$V_t + \tfrac{1}{2} s^{\,2}(t) S^{2p} V_{SS} + r^{01}(t) S V_S - r^{0}(t) V = 0.$$

A simple reduction of the CEV pricing equation to the Bessel equation can be done in several steps. First, we rewrite the equation in terms of forwards and stretched time to maturity

$$S \to F = \exp\left\{\int_t^T r^{01}(t')\,dt'\right\} S, \qquad V \to W = \exp\left\{\int_t^T r^{0}(t')\,dt'\right\} V,$$

$$t \to V = \int_t^T s^{\,2}(t') \exp\left\{2(1-p)\int_{t'}^T r^{01}(t'')\,dt''\right\}.$$

As a result we get

$$W_V - \tfrac{1}{2} F^{2p} W_{FF} = 0.$$

We introduce $X = F^{1-p}/|1-p|$ and $U = X^{-1/2(1-p)}W$ and obtain the Bessel pricing equation:

$$U_V - \tfrac{1}{2}\left(U_{XX} + \frac{1}{X}U_X - \frac{n^2}{X^2}U\right) = 0, \quad n = \frac{1}{2(1-p)}.$$

which can be solved via familiar methods.

Example: CEV Barrier Options via Combination of Reductions and Eigenfunction Expansions

A reduction to the Bessel equation can be used in order to price barrier options only in the simplest case of zero interest rate differential (Linetsky & Davidov 2000, Lipton 2000). In general, we should use either a reduction to Whittaker's equation (Linetsky & Davidov 2000), or to Kummer's equation (Lipton & Little 2000). Assume that interest rates are equal and consider the up-and-out barrier call. The initial condition is

$$U(0,X) = \left(X^{n} - KX^{-n}\right)_{+},$$

the boundary condition is

$$U(t,B) = 0.$$

The corresponding solution can be expanded in the Fourier-Bessel series:

$$U(t,X) = \sum_{m=1}^{\infty} \exp\left\{-\frac{g_{n,m}^{2} t}{2B^{2}}\right\} a_m J_n\left(\frac{g_{n,m} X}{B}\right)$$

Here J_n is the Bessel function, $g_{n,m}$ are its zeroes, and a_m are coefficients which, for the call option can be computed explicitly. Five to ten terms usually give a very accurate approximation. When interest rates are different the solution can be expanded in the Fourier-Kummer series.

The method of eigenfunction expansions has interesting applications in multi-factor problems (Lipton & Little 2000, Little's talk at Math Week Conference).

Stochastic Volatility Models

To explain the smile it is useful to assume that the volatility is stochastic and to choose a model describing its behavior. Several choices are popular (Hull&White, Wiggins, Scott, Stein & Stein, Heston, Lewis):

$$\frac{ds_t}{s_t} = a\,dt + g\,dW_t,$$

$$\frac{ds_t}{s_t} = (a - bs_t)dt + g\,dW_t,$$

$$ds_t = (a - bs_t)dt + g\,dW_t,$$

$$ds_t = \left(\frac{a}{s_t} - bs_t\right)dt + g\,dW_t,$$

$$\frac{ds_t}{s_t} = (a - bs_t^2)dt + gs_t\,dW_t.$$

Usually, it is more convenient to choose a model for the evolution of variance $v_t = s_t^2$.

In the opinion of the speaker, the adequate variance evolution model for forex markets is rather involved, however, for simplicity, we restrict ourselves to the mean-reverting square-root model with time-dependent coefficients:

$$dv_t = k(q - v_t)dt + e\sqrt{v}\,dW_t.$$

European Calls via Fourier Transform

Reduce the pricing problem to the simplest possible form

$$U_t - \tfrac{1}{2} vU_{XX} - e r vU_{Xv} - \tfrac{1}{2} e^2 vU_{vv} - k(q - v)U_v + \tfrac{1}{8} vU = 0,$$
$$U(0, X) = \left(e^{X/2} - e^{-X/2}\right)_+.$$

We solve this problem via the Fourier transform.

It is well-known that the Fourier transform of Green's function has the affine form:

$$G(t, X' - X, v) = \frac{1}{2p} \int_{-\infty}^{\infty} \exp\left[ik(X' - X) + A(t, k) - B(t, k)v\right] dk$$
$$= \frac{1}{2p} \int_{-\infty}^{\infty} \exp\left[ikX'\right] Z(t, k, -X, v) dk \ .$$

Although the Fourier transform of $e^{hX} q(X)$ is difficult to define, it is easy to compute:

$$\int_0^{\infty} e^{(ik+h)X} dX = -\frac{1}{(ik+h)} + \begin{cases} 0, & h < 0, \\ p(k), & h = 0, \\ 2p\, d(k - ih), & h > 0. \end{cases}$$

Thus, we can use Parseval's formula for pricing:

$$U(t, X, v) = \frac{1}{2p} \int_{-\infty}^{\infty} Z(t, k, -X, v) \left(-\frac{1}{ik + 1/2} + 2p\, d(k - i/2) + \frac{1}{ik - 1/2} \right) dk$$
$$= Z(t, i/2, -X, v) - \frac{1}{2p} \int_{-\infty}^{\infty} \frac{Z(t, k, -X, v)}{k^2 + 1/4} dk = e^{X/2} - \frac{1}{p} \int_0^{\infty} \frac{Z_R(t, k, -X, v)}{k^2 + 1/4} dk.$$

We use a similar technique for pricing barrier options.

Options On Volatility

Consider now options on volatility and related issues. These options are path-dependent and discretely sampled by their very nature. There are several definitions of the realized volatility for a given forex evolution scenario. The most popular one is as follows:

$$s_R^2 = \frac{c}{N} \sum_{n=0}^{N-1} \left(\frac{S_{n+1} - S_n}{S_n} \right)^2 .$$

Here N is the number of observations and c is an appropriate constant. The payoff of a typical product is $F(s_R)$. Especially popular choices are

$$F(s_R) = Notional\left(s_R^2 - K\right) \qquad (\text{var}\, swap),$$

$$F(s_R) = Notional\left(s_R - K\right) \qquad (vol\ swap),$$

$$F(s_R) = Notional\left(\max\left\{s_R^2, K_1\right\} - K_2\right) \quad (capped\ \text{var}\, swap),$$

$$F(s_R) = Notional\left(\max\left\{s_R, K_1\right\} - K_2\right) (capped\ vol\, swap),$$

$$F(s_R) = Notional\left(s_R^2 - K\right)_+ \qquad (\text{var}\ \ swaption),$$

$$F(s_R) = Notional\left(s_R - K\right)_+ \qquad (vol\ \ swaption),$$

etc. More generally, we can introduce covariance and correlation swaps which are particularly popular in the forex markets.

Closely related products which are popular in the market place are called cliquettes. Depending on contractual details the corresponding payoffs have one of the two forms

$$\left(S_T - S_{T_1}\right)_+, \qquad Notional\left(S_T / S_{T_1} - 1\right)_+.$$

In order to price options on volatility or cliquettes we can start with the exact definition of s_R, choose a certain model for the evolution of S_t (be it the geometrical Brownian motion or some more elaborate model with deterministic volatility) and estimate the impact of discrete sampling on the price. For instance, in order to price a cliquette in the framework of the deterministic volatility model characterized by a given local volatility $s\,(t,S)$

$$\frac{dS_t}{S_t} = \left(r^0 - r^1\right)dt + s\,(t,S_t)\,dW_t,$$

we can augment the above equation with the auxiliary equation for

$$I_{0,t} = \int_0^t d(V - T_1) S_V dV = \begin{cases} 0, & 0 \le t < T_1 \\ S_{T_1}, & T_1 \le t < T \end{cases},$$

$$dI_{0,t} = d(t - T_1) S_t dt,$$

and solve the two-dimensional pricing equation directly (Little *et al. 2000)*. The terminal conditions are

$$\Phi(t,x) = (1 - x)_+, \qquad \Phi(t,x) = (1 - x)_+ / x, \qquad x = I / S.$$

Needless to say that this approach can only be used for the simplest class of products and fails for more complex (and more important) structures such as volatility swaps.

Alternatively, we can use the continuous approximation for the realized volatility:

$$s_R^2 = \frac{1}{T}\int_0^T s_V^2 dV,$$

and develop the pricing theory based on this approximation.
Some researchers, however, go further and model s_R at maturity directly.
For simplicity below we use the continuous definition of s_R.

As always, it is useful to introduce the augmented system of SDEs for $F_t, v_t, I_{0,t} = \int_0^t v_V dV$:

$$\frac{dF_t}{F_t} = \sqrt{v_t}\, dW_t^F,$$

$$dv_t = k(q - v_t)dt + e\sqrt{v_t}\, dW_t^v,$$

$$dI_{0,t} = v_t dt,$$

The standard degenerate three-dimensional pricing equation written in terms of forward price $V(t, F, v, I)$ of a claim on realized volatility has the form:

$$V_t + \tfrac{1}{2} v F^2 V_{FF} + e r v F V_{Fv} + \tfrac{1}{2} e^2 v V_{vv} + k(q - v)V_v + v V_I = 0.$$

Typical payoffs are:

$$V(T, F, v, I) = I - K,$$

$$V(T, F, v, I) = (I - K)_+, \; etc.$$

The pricing equation has a particular time-independent solution:

$$V^{part}(t, F, v, I) = 2\ln F + I.$$

This solution can be used in order to shift the valuation burden from variance swaps to log contracts and *vice versa.* Specifically,

$$V^{\text{var}\, swap} = V^{part} - V^{\log cont}.$$

The payoff of the log contact has the form

$$V^{\log cont}(T, F, v, I) = 2\ln F + K.$$

The pricing equation for the log contract has the form

$$V_t + \tfrac{1}{2} vF^2 V_{FF} + e r vFV_{Fv} + \tfrac{1}{2} e^2 vV_{vv} + k(q - v)V_v = 0.$$

It is a non-degenerate two-dimensional equation. The corresponding solution is

$$V^{\log cont}(t,F,v,I) = 2\ln F + K + a(t) - b(t)v,$$
$$a(t) = \frac{1}{k}q\left(1 - e^{-k(T-t)} - k(T-t)\right),$$
$$b(t) = \frac{1}{k}\left(1 - e^{-k(T-t)}\right).$$

Accordingly, the price of the variance swap is

$$V^{\mathrm{var}\,swap}(t,F,v,I) = I - K - \frac{1}{k}q\left(1 - e^{-k(T-t)} - k(T-t)\right) + \frac{1}{k}\left(1 - e^{-k(T-t)}\right)v.$$

This price is independent of F.

At inception

$$V^{\mathrm{var}\,swap}(0,F,v,0) = -K - \frac{1}{k}q\left(1 - e^{-kT} - kT\right) + \frac{1}{k}\left(1 - e^{-kT}\right)v.$$

The fair strike is

$$K^* = -\frac{1}{k}q\left(1 - e^{-kT} - kT\right) + \frac{1}{k}\left(1 - e^{-kT}\right)v.$$

The following identity is well-known and rather useful:

$$\ln\left(\frac{S_0}{S}\right)+\frac{S-S_0}{S_0}+\frac{K^*}{2}=\int_0^{S_0}\frac{P(0,S,T,K)}{K^2}dK+\int_{S_0}^{\infty}\frac{C(0,S,T,K)}{K^2}dK.$$

In particular,

$$\frac{K^*}{2}=\int_0^{S_0}\frac{P(0,S_0,T,K)}{K^2}dK+\int_{S_0}^{\infty}\frac{C(0,S_0,T,K)}{K^2}dK.$$

In order to price more general options on realized volatility, we have to find prices of Arrow-Debreu volatility claims which we denote by $G(t,v,I,I')$. These claims solve the problem

$$G_t+\tfrac{1}{2}e^2vG_{vv}+k(q-v)G_v+vG_I=0,$$
$$G(T,v,I)=d(I-I').$$

Thus, we have to solve a degenerate two-factor equation. By homogeneity, it can be solved via the Fourier transform. The corresponding solution has the form

It can be evaluated via the FFT and used in order to price efficiently all options on volatility.

Other Exotic Options on Assets with Stochastic Volatility

Very little is known about exact pricing of exotic options on assets with stochastic volatility. Lipton (1997) showed how to price barrier options. Here we present his derivation and extend it in order to price lookback and passport options on assets with stochastic volatility. Approximate solution of the valuation problem for passport options is given by Henderson & Hobson (2000). This problem is closely related to the problem of optimal choice of hedging strategies for options on assets with stochastic volatility.

For simplicity we assume that interest rates are zero, and Wiener processes driving the underlying and its volatility are uncorrelated.

The general case can be studied as well but the necessary methods are complex and go well beyond the scope of this presentation.

For the double-barrier problem the pricing problem has the form

$$U_t - \tfrac{1}{2} v U_{XX} - e r v U_{Xv} - \tfrac{1}{2} e^2 v U_{vv} - k(q - v) U_v + \tfrac{1}{8} v U = 0,$$

$$U(t, a, v) = U(t, b, v) = 0, \qquad U(0, X, v) = \left(e^{X/2} - e^{-X/2}\right)_+.$$

We can represent its solution as a Fourier series (similar to the no-barrier integral representation):

$$U(t, X, v) = \sum_{n=1}^{\infty} \exp[A(t, k_n) - B(t, k_n) v] y_n \sin[k_n (X - a)],$$

$$k_n = \frac{pn}{b - a}, \quad y_n = \frac{2\left[(-1)^{n+1} k_n \left(e^{b/2} - e^{-b/2}\right) + \sin(k_n a)\right]}{\left(k_n^2 + 1/4\right)(b - a)}.$$

Not surprisingly, passport and lookback options can be studied by the same token. For passport option the pricing problem has the form:

$$V_t + \frac{1}{2} v S^2 \max_Q \left(V_{SS} + 2 Q V_{S\Pi} + V_{\Pi\Pi}\right) + \frac{1}{2} e^2 v V_{vv} + k(q - v) V_v = 0,$$

$$V(T, S, v, \Pi) = \Pi_+.$$

This problem has a homogeneous solution which is governed by

$$\Phi_t - \frac{1}{2} v \max_Q \left(x^2 - 2Qx + 1\right) \Phi_{xx} - \frac{1}{2} e^2 v \Phi_{vv} - k(q - v) \Phi_v = 0,$$

$$\Phi(0, x, v) = x_+.$$

The corresponding $Q = -signx$. Spitting is applicable. The odd component $\Phi^O = \frac{1}{2}x$. The even component solves the problem of the form

$$\Phi^E{}_t - \frac{1}{2}v(x+1)^2\Phi^E{}_{xx} - \frac{1}{2}e^2 v\Phi^E{}_{vv} - k(q-v)\Phi^E{}_v = 0,$$

$$\Phi^E(0,x,v) = \frac{1}{2}x, \qquad \Phi^E{}_x(t,0,v) = 0.$$

Introducing the new spatial variable $X = \ln(1+x)$, and a new dependent variable $U = e^{-X/2}\Phi^E$, we rewrite the pricing problem as

$$U_t - \frac{1}{2}vU_{xx} - \frac{1}{2}e^2 vU_{vv} - k(q-v)U_v + \frac{1}{8}vU = 0,$$

$$U(0,X,v) = \frac{1}{2}\left(e^{X/2} - e^{-X/2}\right), \qquad U_X(t,0,v) + \frac{1}{2}U(t,0,v) = 0.$$

We use the same trick as before and introduce $W = U_X + U/2$. It solves the same pricing equation supplied with the conditions

$$W(0,X,v) = \frac{1}{2}e^{X/2}, \qquad W(t,0,v) = 0.$$

The corresponding problem can be solved via the reflection principle since it does not involve first derivatives with respect to *X*. Once *W* is found, *U* can be found by a simple integration.

Practicalities

Since deviations from the Black-Scholes framework have profound impact on pricing and hedging of exotic derivatives, finding a unique set of ‘correct’ prices is impossible. Because of that practical valuation methods based on the idea of immunization become very important. Usually one tries to create a portfolio consisting of one exotic, several vanilla options, and the underlying which has flat Greeks including Delta, Gamma, Vega, Dvega/Dvol, and Dvega/Dspot.

Conclusions

- Pricing options on the trading account, studying distributions of P&L, and solving the portfolio optimization problem are intimately connected tasks.
- Deviations from the Black-Scholes framework strongly affect prices of exotic derivatives, the impact of static smile is much weaker that the impact of stochastic volatility.
- There exist relatively simple formulas for pricing exotics on assets with stochastic volatility when Wiener processes driving the underlying and its volatility are uncorrelated.
- There exist very complex formulas for pricing exotics on assets with stochastic volatility when Wiener processes driving the underlying and its volatility are correlated.

Further reading

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