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Valuing Options on Baskets of Stocks and Forecasting the Shape of Volatility Skews

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Outline

- Extract risk-neutral distribution under maximal uncertainty.
- Estimating risk-neutral distributions from an one-parameter family of distance measures between probability distributions.
- Deriving risk-neutral distributions using extended power utility functions.
- Is the implied volatility skews of index options justified by historical data?
- Ranking equity options using strike-adjusted spread.
- Valuing options on basket of stocks.
- Forecast the shape of smile and end-of-day mark to market.
- Summary

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References:

M. Stutzer, "*A Simple Nonparameteric Approach to Derivative Security Valuation*" J. of Finance, 51, 1996.

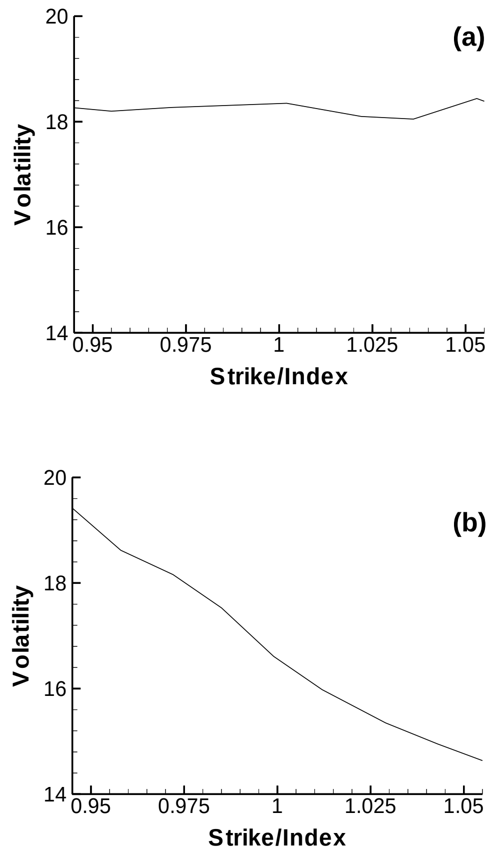
Emanuel Derman and Joe Zou, "*Is the Volatility Skew Fair?*" Goldman Sachs Quantitative Strategies Research Notes, 1997.

Joe Zou and Emanuel Derman, "*Strike-adjusted Spread: A New Metric For Estimating The Value of Equity Options*". Goldman Sachs Quantitative Strategies Research Notes, 1999.

Joe Zou and Emanuel Derman, "Monte Carlo Valuation of Path Dependent Options On Indexes with a Volatility Smile". J. of Financial Engineering, V6, 1997.

Motivation

Representative implied volatility skews of S&P 500 options. (a) Pre-crash. (b) Post-crash.



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Can the volatility skew be extracted from realized historical returns?

For a given stock or stock index, how is an investor to know which strike and expiration provides the best value?

What metric can options investors use to gauge their estimated excess return?

What is the appropriate volatility surface for an illiquid basket?

Suppose an investor is interested in buying a 10% out-of-the-money put and selling a 10% out-of-the-money call on a basket of bank stocks. The basket consists of equal number of shares of 5 stocks: Bank One, Chase, JP Morgan, Wells Fargo and Bank of America.

Bank Basket = ONE + CMB + JPM + WFC + BAC

Can we forecast the shape of a skew curve when one of the options' implied volatility has changed?

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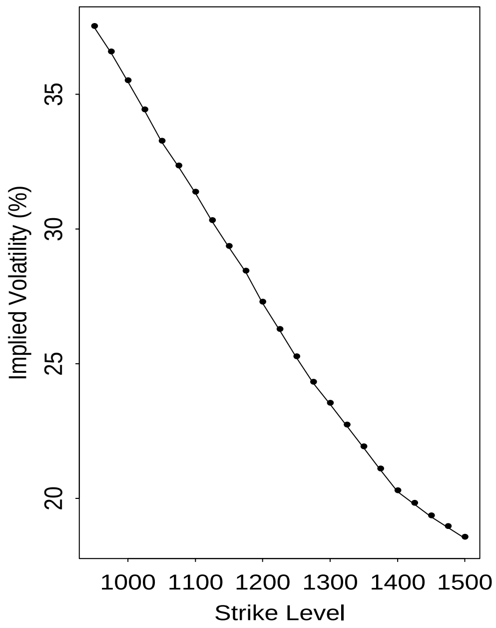
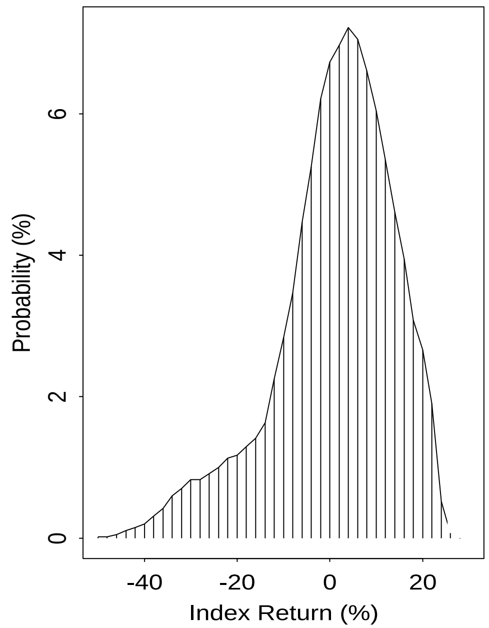
Option Prices Implied Distribution

Implied Volatility Skew ⟷ Skewed Distribution

(S&P 500 index option implied three-month distribution and the corresponding implied volatility skew as of 3/10/99)





$$C_{K,T}(S, \Sigma_{K,T}) = e^{-r(T-t)}\int Max(S_T - K, 0)Q(S_T)dS_T$$

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Empirical Distribution

- Step 1: get realized stock price time series.
- Step 2: estimate the rolling T-period return histogram and calculate the empirical distribution of the stock returns

$$R_t = \log\left(\frac{S_t}{S_{t-T}}\right)$$

If R falls in the jth bin, increase the count in the jth bin by one. The final count in the bin is a good estimate of the probability $P(R_j)$.

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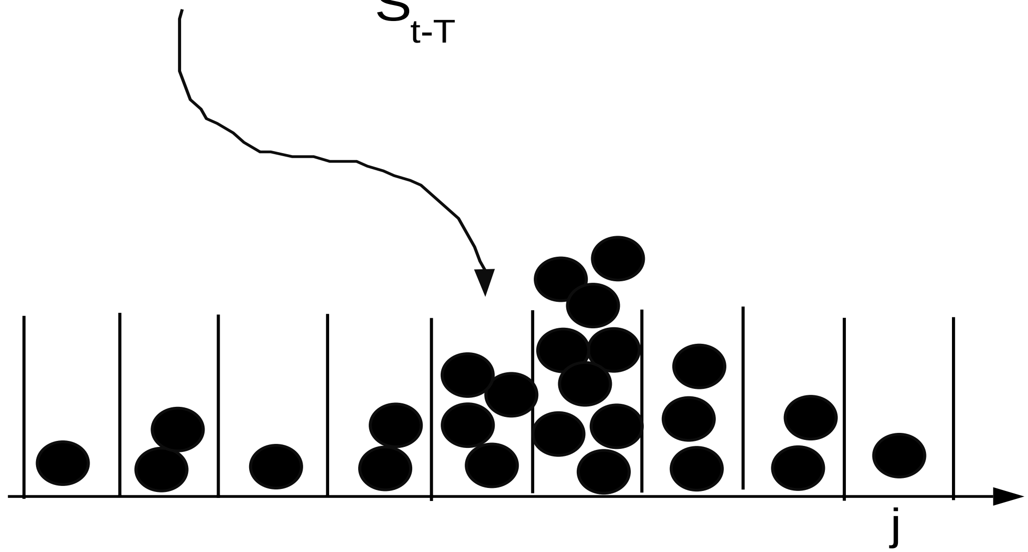
Histogram and empirical distribution

$$R = \log\left(\frac{S_t}{S_{t-T}}\right)$$



T-period Returns

Probability ∝ "bean counts" in the return bins.

$$\sum P(j) = 1$$

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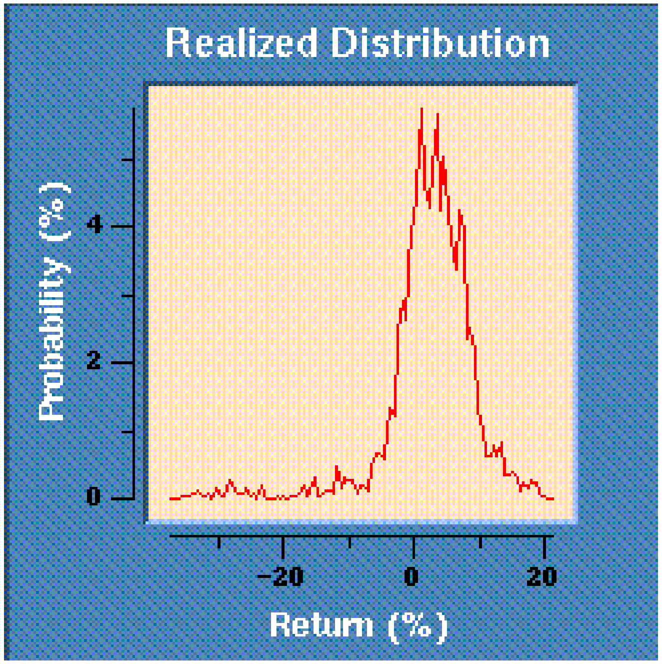
S&P 500 Three-Month Empirical Distribution



Can we use the empirical distribution to estimate the risk-neutral distribution?

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Uncertainty and Market Equilibrium

Consider two binomial distributions:

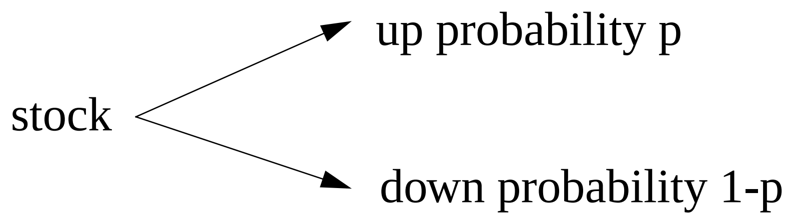
- X is far more predictable than Y,
- X is not stable (more bulls than bears),
- Y is most uncertain about the future market movements,
- Y is more stable (equal number of bears and bulls) and
- Y is likely to be the market equilibrium distribution!

The equilibrium distribution tends to be far less certain about the direction of future market movement.

Information and Probability

- Probability measures the uncertainty about a single random event
- Entropy measures the uncertainty of a collection of random events.

Consider a stock whose next move may be up or down:



Information conveyed by an up move: $I(up) = -\log p$

Information conveyed by a down move: $I(down) = -\log(1-p)$

If *p=1*, then an up-move conveys no information at all!

If everyone expected the stock to go up, and it actually moves down, the outcome is more informative.

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Entropy and Probability Distribution

It is the expected amount of information of all possible outcomes

$$S(p) = -[p \log p + (1-p)\log(1-p)]$$

This entropy is maximized if *p*=50%:

Maximum Entropy $\Leftrightarrow$ Maximum Uncertainty

The risk-neutral distribution we are seeking contains more states than the simple up and down states and thus is more complicated than the example. But the basic idea is similar.

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Extract Risk-Neutral Distribution by Minimizing Relative Entropy

Maintain maximum uncertainty while satisfying the forward condition.

$$\text{Relative-Entropy Function:}\quad S(P, Q) = \sum_j Q(R_j)\log\left(\frac{Q(R_j)}{P(R_j)}\right)$$

Minimize *S(P,Q)* subject to the forward condition:

$$S_0 e^r = \sum_j S_0 e^{R_j} Q(R_j)$$

The minimum relative entropy method will change the shape of the prior distribution in the least prejudicial way so as to satisfy the forward condition.

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The solution to the minimization problem is:

$$Q_\lambda(S_0, 0; S_T, T) = \frac{P(S_0, 0; S_T, T)}{\int P(S)\exp(-\lambda S)dS}\exp(-\lambda S_T)$$

where the constant λ can be found numerically by solving the forward constraint

$$\int Q_\lambda(S_0, 0; S_T, T)S_T dS_T = S_0 e^{rT}$$

where P is a given prior distribution.

An One-parameter Family of Distance Measures Between Probability Distributions.

Reference: Huaiyu Zhu, "Bayesian Geometric Theory of Learning Algorithms", Santa Fe Institute Working paper, 1995.

Consider the following *Information Deviation* with parameter $\delta \in (0, 1)$

$$S_\delta(P, Q) = \int \frac{\delta p + (1-\delta)q - p^\delta q^{1-\delta}}{\delta(1-\delta)} ds$$

The deviation S_0 and S_1 are defined as limits as $\delta \to 0$ and $\delta \to 1$

$$S_1(P, Q) = S_0(Q, P) = \int p \log\left(\frac{p}{q}\right) ds$$

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It's straightforward to show that $S_\delta(P, Q)$ has the properties of square distances:

$$S_\delta(P, Q) \geq 0$$

$$S_\delta(P, Q) \equiv 0 \Leftrightarrow P \equiv Q$$

$$S_\delta(aP, aQ) = aS_\delta(P, Q)$$

$$S_\delta(P, Q) = S_{1-\delta}(Q, P)$$

Minimizing $S_\delta(P, Q)$, subject to the risk-neutral constraint yields

$$Q_\delta(s|s_0) = P(s|s_0)[c_0(\delta) + c_1(\delta)s]^{-1/\delta}$$

where the constants c_0 and c_1 for a given δ solve the following constraints:

$$\int P(s|s_0)[c_0 + c_1 s]^{-1/\delta} ds = 1$$

$$\int sP(s|s_0)[c_0 + c_1 s]^{-1/\delta} ds = s_0 e^r$$

In the next section, we derive the risk-neutral distribution using an derivatives asset allocation model based on a class of extended power utility functions: (See Robert C. Merton)

$$U(W) = \frac{\gamma}{1-\gamma}\left[\left(a + \frac{b}{\gamma}W\right)^{1-\gamma} - 1\right]$$

where $\gamma \neq 0$, $b > 0$, and $a = 1$ if $\gamma \to \infty$ (corresponding to the exponential utility). The limit as $\gamma \to 1$ leads to the log-utility. It can be shown that the risk-neutral density obtained using the utility function with exponent γ, is the same as Q_δ with $\gamma = 1/\delta$.

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Derivatives Asset Allocation and Risk-Neutral Distributions Based on Extended Power Utility Functions

Consider an economy in equilibrium:

- A representative investor with initial wealth W_0
- The investor has a market view expressed through a conditional density $P(S_0, 0; S_T, T)$
- An Arrow-Debreu security with parameter E has a price given by

$$\pi(S, t; E, T) = DQ(S, t; E, T)$$

where $D = 1/(1 + r_f)$ is the discount factor, and $Q(S, t; E, T)$

is the risk-neutral density.

- Let α be the portion of wealth allocated to riskless bonds and $(1\text{-}\alpha)$ be the portion allocated to risky assets.
- Let $\omega(E)dE$ be the portion of the $(1\text{-}\alpha)$ that is invested in Arrow-Debreu security with parameter E.
- At the end of period T, the total investor wealth will be

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$$W_T(S_T) = W_0[1 + \alpha r_f + (1-\alpha)\int \omega(E) r_E(S_T) dE]$$

where

$$r_E(S_T) \equiv \frac{\pi(S_T, T; E, T) - \pi(S_t, t; E, T)}{\pi(S_t, t; E, T)} = \frac{\delta(S_T - E)}{DQ(S, t; E, T)} - 1$$

and

$$\int dE r(E) Q(S_t, t; E, T) = r_f$$

- If the allocation $\omega(E)dE$ is changed, the supply and demand for the Arrow-Debreu security will also change and thus the shape of the risk-neutral density function $Q(S_t, t; E, T)$ will also change. Therefore, to achieve a market equilibrium, we must solve the asset

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allocation problem by maximizing the expected utility $U(W_T)$:

$$\underset{\{\alpha,\, \omega(E),\, Q(E)\}}{Max} \{E_p[U(W_T)]\}$$

subject to

budget constraint: $\int \omega(E) dE = 1$

normalization: $\int Q(E) dE = 1$

forward constraint: $\int Q(E) E dE = S_0(1 + r_f)$

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Solving the optimization problem

$$Q(S_0, t; S_T, T) = \frac{U'(W_T(S_T))}{E_P[U'(W_T)]} P(S_0, t; S_T, T)$$

where

$$W_T(E) = W_0\left[\alpha(1 + r_f) + (1 - \alpha)\frac{\omega(E)}{DQ(E)}\right]$$

and

$$E_P[U'(W_T)] = \frac{\lambda_1}{W_0(1 - \alpha)r_f}$$

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$$E_P[U'(W_T)r_E(S_T)] = \frac{\lambda_1}{W_0(1-\alpha)}$$

$$\frac{\omega(E)}{Q(E)} = \frac{r_f}{1+r_f}\left[\frac{\lambda_2}{\lambda_1} + \frac{\lambda_3}{\lambda_1}E\right]$$

$$\frac{\lambda_2}{\lambda_1} + \frac{\lambda_3}{\lambda_1}S_0(1+r_f) = \frac{1+r_f}{r_f}$$

We now specialize in exponential utility:

$$U(W_T) = -\exp(-bW_T) \quad \text{with} \quad b > 0$$

The final results:

$$Q(S_t, t; E, T) = P(S_t, t; E, T)\exp(c_0 - c_1 E)$$

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where the constant c_0 and c_1 are to be determined by the normalization and forward price constraints

$$\int Q(S_0, 0; S_T, T) S_T dS_T = S_0 e^{rT}$$

$$\int Q(S_0, 0; S_T, T) dS_T = 1$$

and they are independent of the parameter, b, of the utility function. They only depend on the prior distribution P, and the constraints. The parameter, b, is characteristic of the representative investor's risk aversion. It is essential that the risk-neutral distribution be independent of the investor's risk aversion!

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Finally, the representative investor's allocation of Arrow-Debreu security with parameter E, $\omega(E)$, can be calculated using c_0 and c_1.

The most important feature of this solution to the asset allocation problem is that the risk-neutral probability density function $Q(S_t, t; S_T, T)$ is of the same form given by the minimum relative entropy approach. This is true for the extended power utility, provided that we use the generalized relative entropy with the proper choice of the parameter

$\delta = 1/\gamma$ (the proof follows the same as above).

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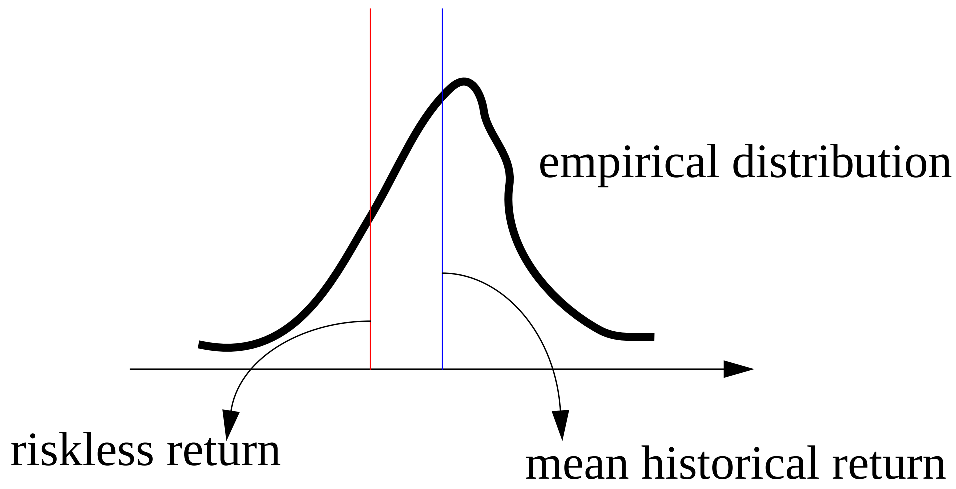
Is the Volatility Skews of Index Options Justified by Historical Data?

- Justified: Fair risk-neutral expected value using the empirical distribution as a prior.



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Calculate the expectation of option's payoff at expiration

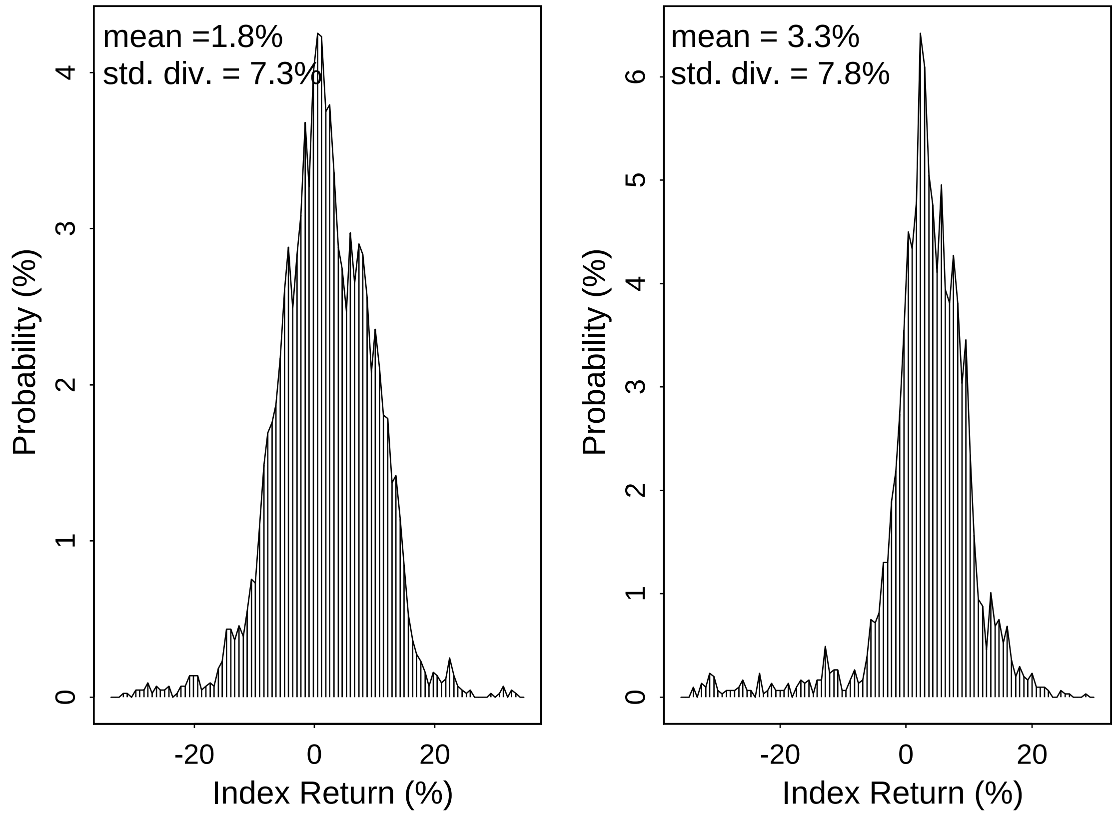
$$E_T[C_K|S] = \sum_j max(Se^{R_j} - K, 0)Q(R_j)$$

Discounting the expectation, and extracting the implied volatility from the Black-Scholes formula.

$$e^{-rT}E_T[C_K|S] = BS(S, K, T, r, d, \Sigma_{K,T})$$

Repeating this for all strikes and maturities, we extract an implied volatility surface from the historical return distribution.

SPX Pre-crash and Post-crash Distributions



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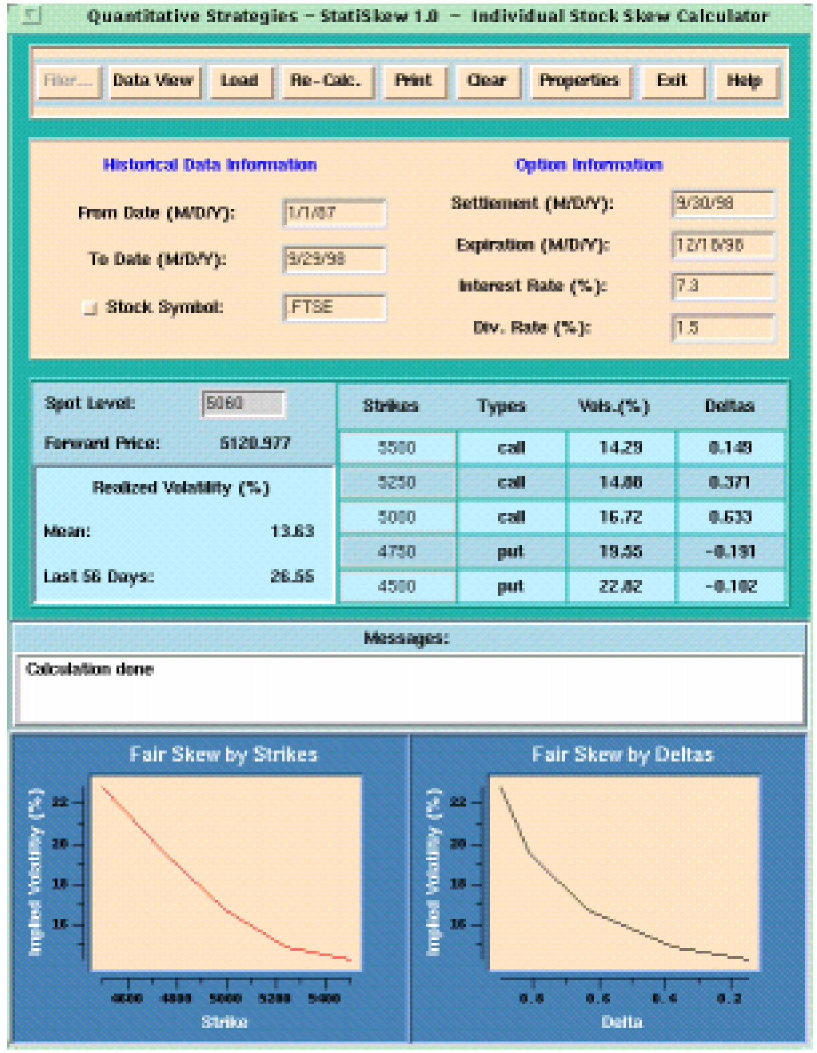
FTSE “Fair” Skew (9/30/98 - 12/18/98)



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Examples

- Applications of our model to S&P 500 index and DAX, and FTSE-100 index show that the slope of the skews are approximately fair!

Note: The absolute levels of implied volatility can vary dramatically over time, the slope of the skew is relatively stable.

Size of skew: compare actual data with model results. (25 delta put-25 delta call)

Index	Normal Spread* (25p -25c)	Recent Spread (25p-25c)	"Fair" Spread (25p-25c)
SPX	4-7%	14%	6.0%
DAX	3-6%	10%	3.5%
FTSE	2-6%.	10%	4.0%

* three-year data from October 95 to September 98 for three month options.

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Strike-Adjusted Spread

The *SAS* of a stock option is calculated as follows.

1. First, choosing some historically relevant period, we obtain the distribution of stock returns over time *T*. This empirical return distribution characterizes the past behavior of the stock.

2. We use the empirical return distribution as a statistical prior to provide us with an estimate of the risk-neutral distribution by minimizing the entropy associated with the difference between the distributions, subject to ensuring that the risk-neutral distribution is consistent with the current forward price of the stock. We call this risk-neutral distribution obtained in this way the **risk-neutralized historical distribution, or RNHD.**

3. We then use the RNHD to calculate the expected values of standard options of all strikes for expiration *T*, and convert these values to Black-Scholes implied volatilities. We denote the Black-Scholes implied volatility of an option whose price is computed from this distribution as Σ_H. This is our estimated fair option volatility.

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4. For an option with strike K and expiration T, whose market implied volatility is $\Sigma(K, T)$, the strike-adjusted spread in volatility is defined as

$$SAS(K, T) = \Sigma(K, T) - \Sigma_H(K, T)$$

This spread is a measure of the current richness of the option based on historical returns.

5. We often use a modified version of SAS for which the risk-neutralized historical distribution is further constrained to reproduce the current market value of at-the-money options. We call this (additionally constrained) distribution the **at-the-money adjusted, risk-neutralized historical distribution**, or $\mathbf{RNHD_{ATM}}$. The strike-adjusted spread computed using this distribution, denoted $SAS_{ATM}(K, T)$, is a measure of the relative value of different strikes, assuming that, by definition, at-the-money-forward implied volatility is fair.

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$$SAS_{ATM}(K, T)|_{K = S_F} = 0$$

Use $SAS_{ATM}(K, T)$ to Rank Equity Options

on the same underlyer, in order to determine which strikes provide the best value by historical standards.

FIGURE 1. (a) Fair and market skews for S&P 500 index options on May 18, 1999. (b) SAS_{ATM} for the same options.
The options considered expire on September 17, 1999. Both fair and market implied volatilities are constrained to match at the money, forward. The RNHD is constructed using returns from May 1987 to May 1999, including the 1987 crash.

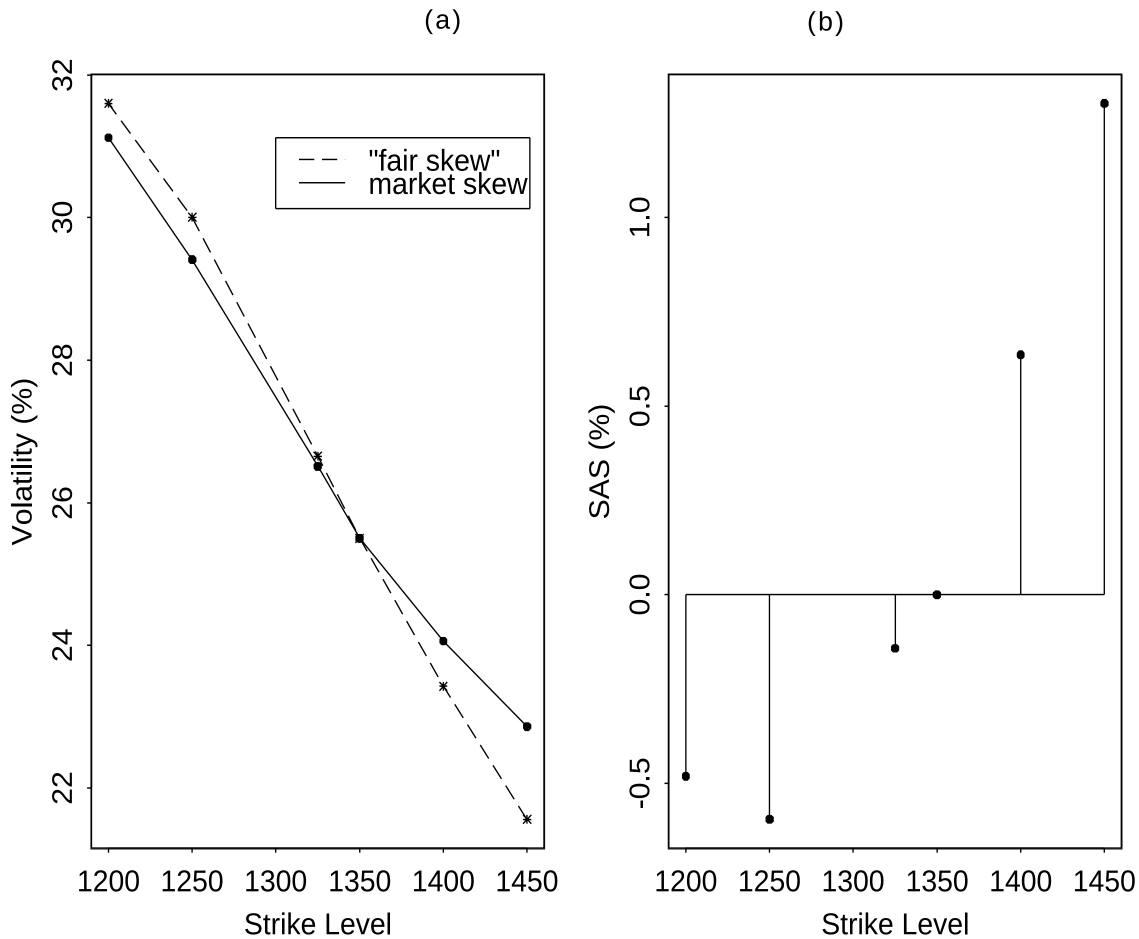

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FIGURE 2. (a) Fair and market skews for S&P 500 index options on May 18, 1999. (b) SAS_{ATM} for the same options.
The options considered expire on September 17, 1999. Both fair and market implied volatilities are constrained to match at the money, forward. The RNHD is constructed using returns from May 1988 to May 1999, thereby excluding the 1987 crash.

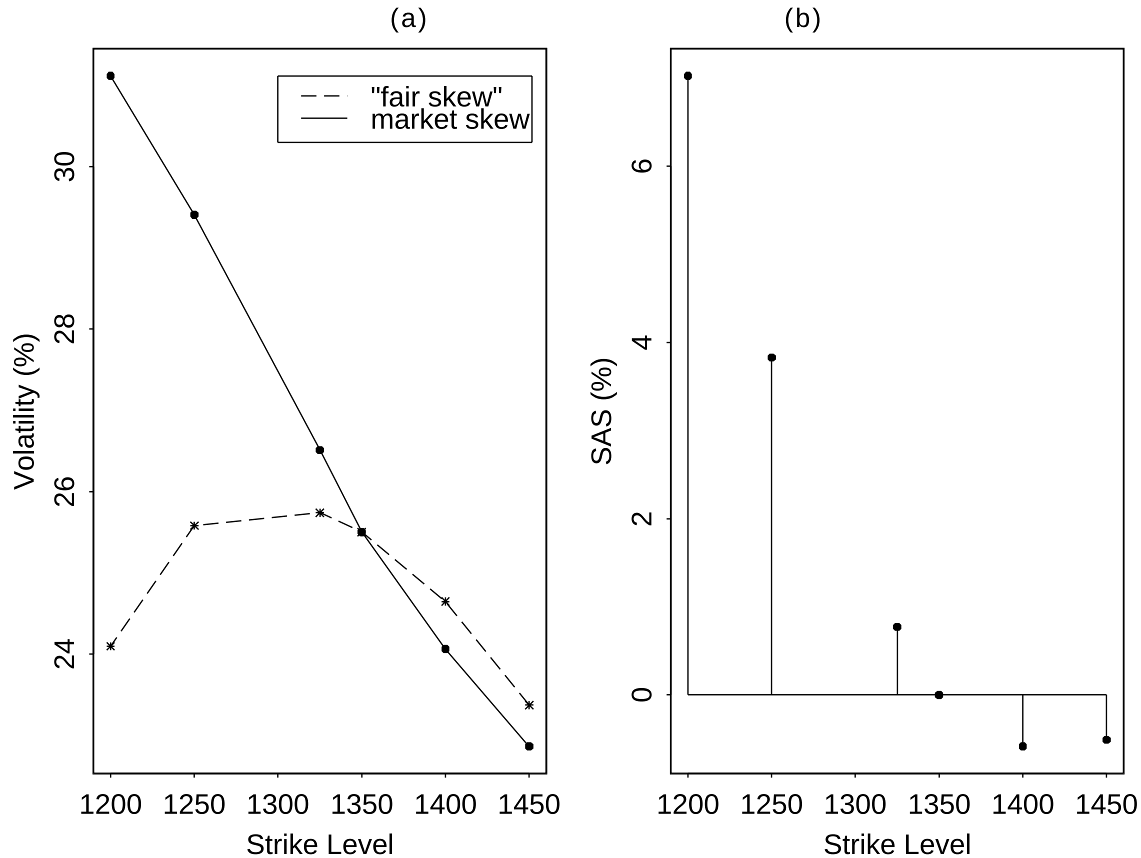
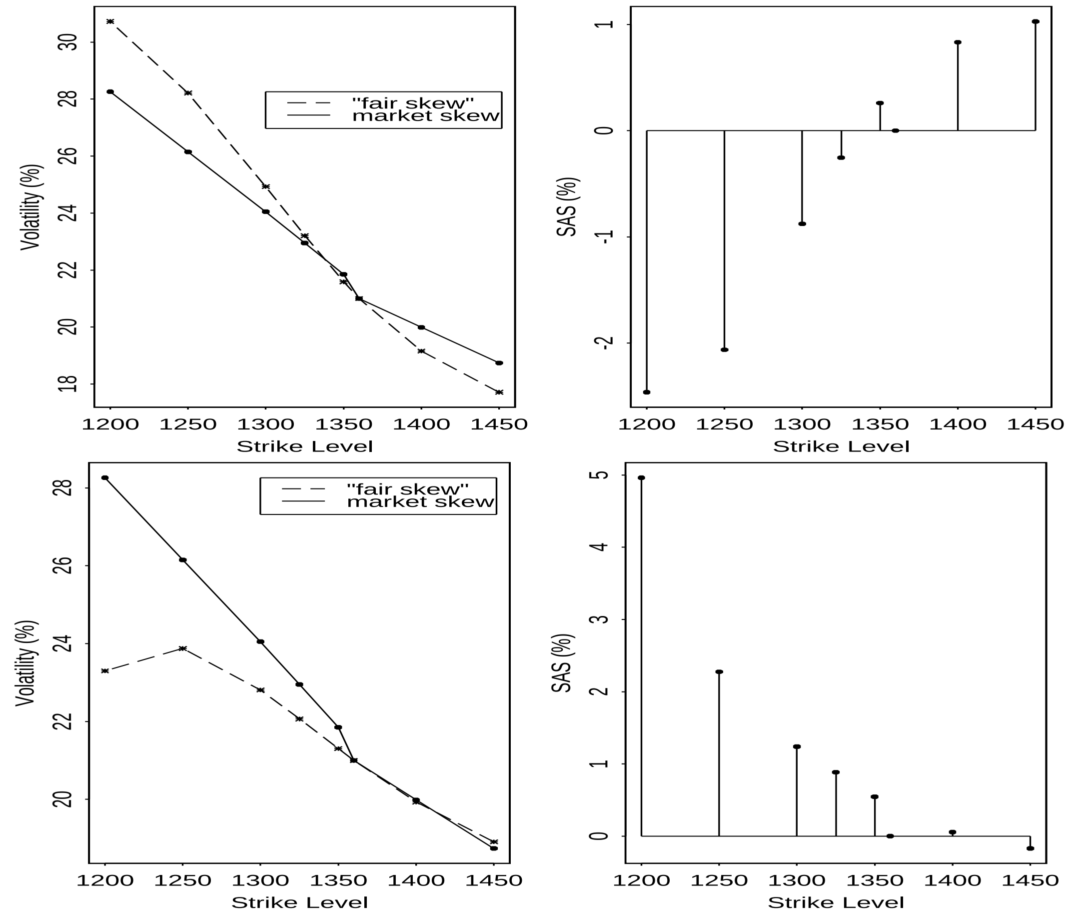


FIGURE 3. Re-evaluated SAS_{ATM} on June 21, 1999 for September 17, 1999 S&P 500 options. The top two figures correspond to the crash-inclusive distributions of Figure 1.; the bottom two correspond to the crash-exclusive distributions of Figure 2..



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Valuing Options on Basket of Stocks

- Estimate Basket Volatility - the old way

$$\sigma_B^2 = \sum_i w_i^2 \sigma_i^2 + 2 \sum_j \sum_{i<j} w_i w_j \rho_{ij} \sigma_i \sigma_j$$

where, w_i is the weight of stock i in the basket, and ρ_{ij} is the correlation between stock i and stock j.

Problems:

- Component stocks usually do not follow the log-normal process as the implied volatility skews show. Neither does the basket.
- The correlations between stocks can vary with the market volatility.
- No obvious way of estimating volatility skews of the basket.

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Correlations and the Market Volatility

Consider two stocks whose returns may be linked to the market return via CAPM:

$$r_1 = \beta_1 r_m + \varepsilon_1$$

$$r_2 = \beta_2 r_m + \varepsilon_2$$

where $(\varepsilon_1, \varepsilon_2)$ are the "tracking errors", and r_m is the market return. It's easy to see the correlation

$$\rho(r_1, r_2) = \frac{\beta_1 \beta_2 \sigma_m^2}{\sqrt{\beta_1^2 \sigma_m^2 + \varepsilon_1^2}\sqrt{\beta_2^2 \sigma_m^2 + \varepsilon_2^2}}$$

increases as a function of the market volatility.

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Basket Volatility Skews - the new way

We are interested in valuing European options on a basket whose spot price is S.

$$C_{K,T}(S,t) = e^{-r(T-t)} E_Q[\text{ Payoff at T} \mid S, t]$$

where Q is the risk-neutral probability distribution whose mean is the riskless interest rate.

The distribution Q is calculated by minimizing the relative entropy using the empirical distribution of the basket as a prior.

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Basket Option Examples

- For the bank basket shown at the beginning, the volatility spread between the 10% OTM call and the 10% OTM put is almost 7 vol. points!
- We compare the market implied skew of BKX index with the skew calculated by our model. The size of skew seen from the market is almost identical to that predicted by our model, even though the actual level of the volatility is off by roughly 5 points.

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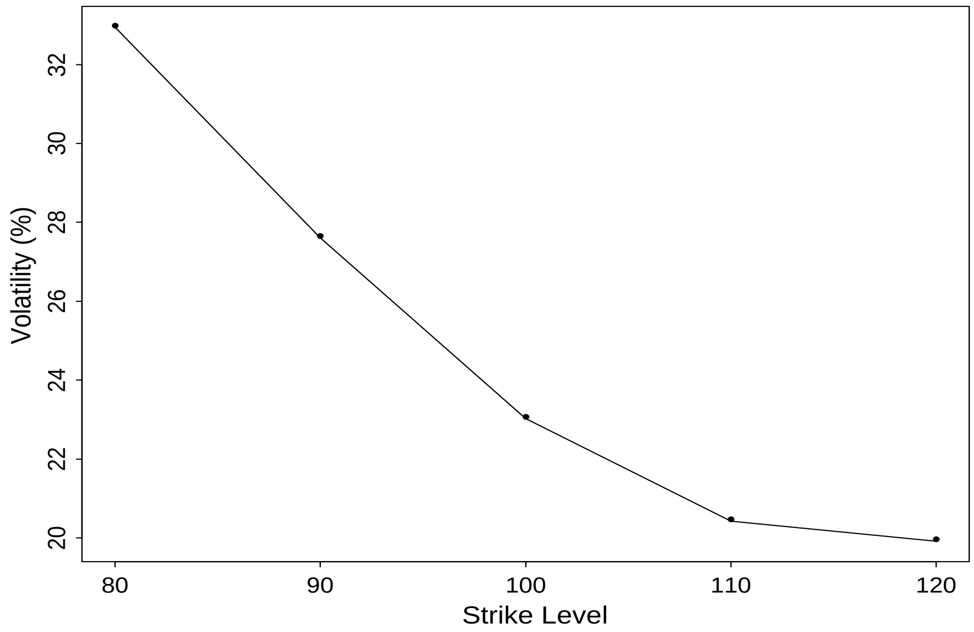
The estimated fair three-month implied volatility skew for the basket of five bank stocks listed in the text, estimated from the risk-neutralized historical distribution using returns from June 1987 to June 1999



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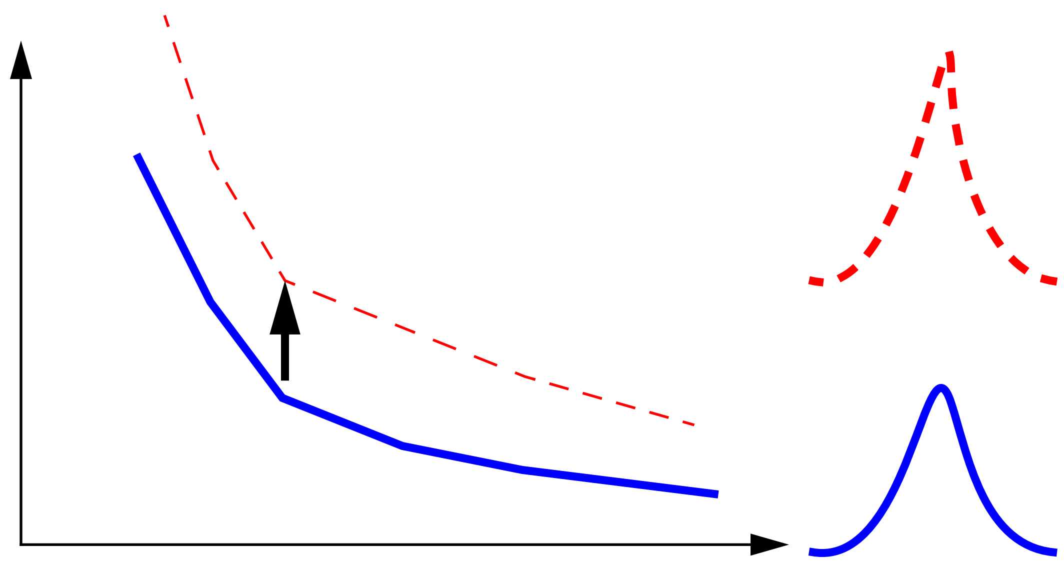
Forecast the Shape of the Skew From A Change in A Single Option Price

- How to adjust quotes for other options on a skew curve given that one of the options has traded away?
- End-of-day mark to market in the presence of stale option prices.



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$$Min\left\{S(Q,\tilde{Q}) = E_{\tilde{Q}}\left[\log\left(\frac{\tilde{Q}(S)}{Q(S)}\right)\right]\right\}$$

$$\text{S. T.} \quad \int \tilde{Q}(S_T) S_T dS_T = S_0 e^{r_f T}$$

$$\tilde{C}(K_j, T) = e^{-r_f T} \int Max[S_T - K_j, 0] \tilde{Q}(S_T, T | S_0, 0) dS_T$$

where Q is the original implied distribution, $\tilde{Q}$ is the forecasted distribution, and $\tilde{C}$ is the new option price for strike K_j.

$$\tilde{Q}(S_T, T; S_0, 0) = \frac{Q(S_T, T; S_0, 0) \exp[-\lambda_1 S_T - \lambda_2 f_j(S_T)]}{\int Q(S) \exp[-\lambda_1 S - \lambda_2 f_j(S)] dS}$$

where $f_j(S) = Max(S - K_j, 0)$.

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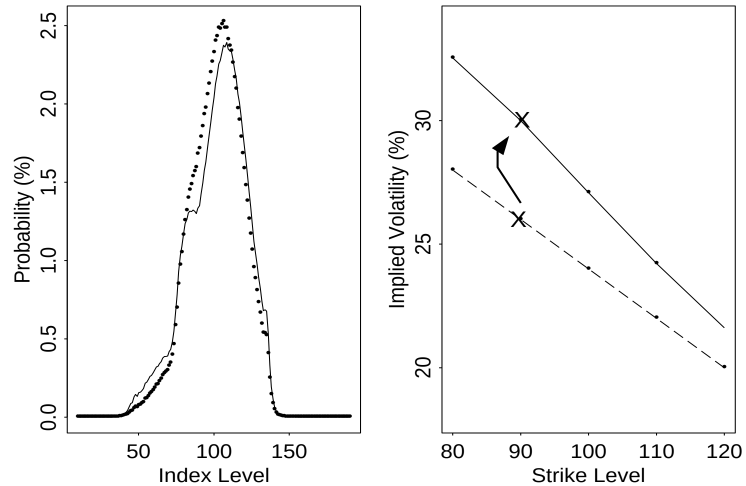
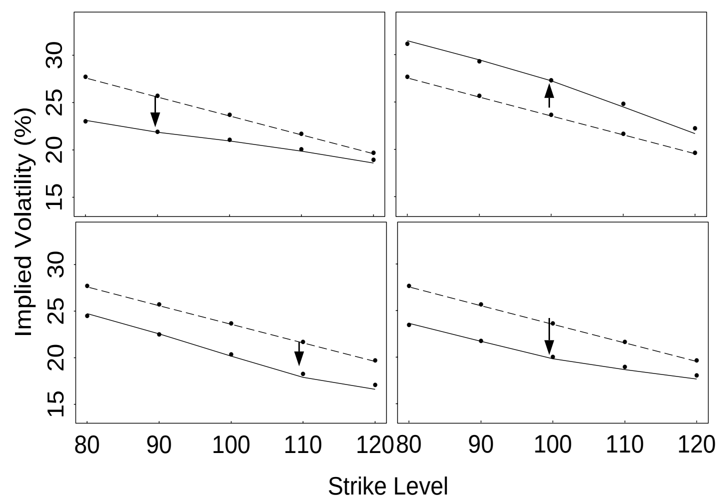
Example: Updated distribution and volatility skews after one volatility on the skew curve has changed



(a) (b)



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Summary:

- For the pre-crash period, our method produces no appreciable skew. For the post-crash period, the model produces significant skew that is comparable with the observed market data.
- Strike-adjusted spread as a gauge of the relative richness of equity options.
- Particularly useful for valuing OTC options on single stock or on a basket of stocks.
- The method may be used to forecast the change of smile when some of the options have traded away.
- It may be helpful for volatility traders to mark to market at the end of trading days with stale option prices.
- An equilibrium asset allocation model of Arrow-Debreu securities with a class of utility functions yields the same risk-neutral distributions as those obtained by minimizing a class of generalized relative entropy function.